Research Statement
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Overall my research concerns mainly *quasilinear hyperbolic systems* of partial differential equations, with a preference to those problems that arise from a physical or geometric context (e.g. general relativity, fluids, or nonlinear elasticity) and to the use of geometrically inspired techniques (e.g. the vector field method). Currently I focus my attention on four inter-related themes.

I. Lorentzian constant mean curvature flow

In classical physics and differential geometry, the surface tension of a fluid interface or a thin film is often modeled by mean curvature, and this leads to the well-known Plateau problem of minimal surfaces immersed in Euclidean space. Its *dynamical* counterpart, representing relativistic membranes evolving under the balance between surface tension and external normal forces, forms the classical foundation to the quantum mechanical theory of extended bodies [3, 30]. As model problems, it is of particular interest to understand the case where the external normal force is constant — be it vanishing, representing the case of freely evolving extended test objects (as opposed to point particles); or signed, for example soap bubbles supported by pressure differentials. We refer to the evolution problem as the Lorentzian constant mean curvature (CMC) flow. Observe that with a global rescaling, the constant normal force can be supposed to have magnitudes $+1$, -1, or 0. The last case is sometimes referred to in the literature as (depending on sign conventions) either minimal, extremal, or maximal time-like hypersurface.

Geometrically we consider a hypersurface M of $(d+1)$-dimensional Minkowski space $\mathbb{R}^{1,d}$ with the property that the induced metric on M is Lorentzian (so that M is time-like) and the mean curvature scalar is a constant. If we choose a coordinate system in which M is expressed locally as a graph $\{x^d = \varphi(x^0, x^1, \dots, x^{d-1})\}$ over a time-like hyperplane, the function φ satisfies the familiar equation

$$-\frac{\partial}{\partial x^0}\left(\frac{\partial_{x^0}\varphi}{\sqrt{1-(\partial_{x^0}\varphi)^2+\sum_{i=1}^{d-1}(\partial_{x^i}\varphi)^2}}\right)+\sum_{j=1}^{d-1}\frac{\partial}{\partial x^j}\left(\frac{\partial_{x^j}\varphi}{\sqrt{1-(\partial_{x^0}\varphi)^2+\sum_{i=1}^{d-1}(\partial_{x^i}\varphi)^2}}\right)=c. \tag{1}$$

Notice that the denominator is well-defined so long as M remains time-like, and the presence of dynamics is exhibited by the *hyperbolic* nature of the system: (1) is clearly a quasilinear wave equation. Thus, reflecting the physical origins of this equation, the natural problem to study is that of its initial value problem. Note that in general, however, M cannot be globally written as the graph over a hyperplane, and the local equation (1) must be replaced by more geometric equations describing the embedding $M \hookrightarrow \mathbb{R}^{1,d}$ if one is interested in the solution in the large.

The study of Lorentzian CMC flow has some technical similarities with the study of the vacuum Einstein equations in general relativity. Both can be reformulated as studying *quasilinear* geometric wave equations or hyperbolic systems; one should contrast this to the well-studied wave-map equations which generate *semilinear* problems. The quasilinearity poses not only challenges in analysis, but also in geometry, since in both Lorentzian CMC flow and general relativity, the principal part of the evolution is governed by the geometric wave operator associated to a Lorentzian manifold that can be very different from the familiar Minkowski space. Lastly, the role played by the constant normal force in the Lorentzian CMC flow is similar to the role played by the cosmological constant in the vacuum Einstein equations.

The ultimate goal in studying Lorentzian CMC flow is to obtain complete understanding of the asymptotic behaviors of solutions to the initial value problem: whether the solution blows up in finite time and how it blows up; whether the solution exists for all time and what are the asymptotic profiles. Existing works are far from completing this goal, and generally fall into three types: the construction of explicit or algebraic solutions or representations satisfying specific properties (e.g. [29]); the description of asymptotic properties under simplifying assumptions such as low dimensionality [65] or additional symmetries [4] (see also my recent preprint [81]); and the investigation of stability properties of special solutions. My interests in Lorentzian CMC flow currently focus on this third type.

Existing stability results

Vanishing mean curvature, $c = 0$ The first explicit results were given for the hyperplane, which is a trivial solution to (1) where the entire solution can be represented as a graph. The hyperplane was shown to be stable by Brendle [5] in higher dimensions $d \geq 4$ and Lindblad [60] in $d \geq 2$. Their results can largely be deduced as special cases of the pioneering work on small data global existence theory of quasilinear wave equations by Christodoulou [8] and Klainerman [42]. The stability mechanism in $d \geq 3$ is the dispersive decay of solutions[1] to the linear wave equation on Minkowski spaces in conjunction with the null condition satisfied by (1).

More recently, Roland Donninger, Joachim Krieger, Jeremie Szeftel, and I made a breakthrough [21] in dealing with the stability problem in the non-graphical case. We studied in $d = 3$ the stationary catenoid solution: recall that the catenoid is an asymptotically flat complete minimal surface embedded in $\mathbb{R}^3$, and so extends as a stationary solution to the Lorentzian CMC flow. Our result, with minor modifications, is applicable also to the $d > 3$ case where the catenoid is replaced by the unique asymptotically flat axially symmetric complete minimal surface embedded in $\mathbb{R}^d$. As the background geometry is very different from the standard Minkowski spaces, neither the conformal method [8] nor the vector field method [42] can be applied directly to analyze the associated equations, whose linearization is a wave equation on a catenoid background with a potential term. Furthermore, this system turns out to be linearly unstable due to the variational instability of the catenoid as a minimal surface [25]. Our main theorem can be roughly summarized as the existence of a center-stable manifold.

Theorem 1 (DKSW 2013)**.** *Under axial symmetry, in a neighborhood of the catenoid initial data there exists a codimension 1 Lipschitz manifold, transverse to the linearly unstable mode, such that initial data on this manifold give rise to solutions converging back toward the catenoid.*

To overcome the difficulties, the dispersive decay estimates for the linearized equations were proven using a distorted Fourier transform to account for the geometry and the potential term, and to take advantage of the asymptotic flatness of the catenoid. This is coupled to a careful choice of coordinates for the perturbation equations which exhibits the null structures within the nonlinearities. The assumption of axial symmetry is technical: it turns out the geometry of the catenoid holds a second obstruction, in that it exhibits trapped geodesics. It is well-known that such geometries complicates the availability of dispersive estimates, so in [21] we postulate axial symmetry to avoid dealing with this issue.

Positive mean curvature, $c = +1$ To my knowledge my recent paper [81] is the first to consider the stability property of solutions to these equations; my result mainly concerns the full nonlinear

[1] When $d = 2$, (1) is a quasilinear wave equation in $1 + 1$ dimension, and the linearized equation has no decay. Lindblad's result [60] strongly depends on Christodoulou's conformal method [8] and the highly special algebraic structure of (1).

stability of a family of *expanding* (hence not stationary) solutions that includes the de Sitter space embedded in $\mathbb{R}^{1,d}$ as the hyperboloid $\{-(x^0)^2 + (x^1)^2 + \ldots + (x^d)^2 = 1\}$. The expanding background offers its own challenges. A direct linear analysis shows that the system is unstable. Furthermore, one can show that this instability is *infinite dimensional*. In turns out that this instability can be interpreted in two ways, both tied to the geometry. On the one hand, one sees this as a phantom instability due to a poor gauge choice, in that the perturbed solution is inappropriately compared to the background solution. Since the background is expanding, this comparison error also grows. On the other hand, the expanding solutions contain what are called in the relativity literature *cosmological horizons*, which allow solutions to develop "spatially localized features". This has the effect of amplifying the finite dimensional instability (which we expect) due to the symmetries of Minkowski space to an infinite dimensional family of instabilities. Understanding the nature of the apparent instability leads to the following *stability* theorem.

Theorem 2 (W 2014)**.** *Expanding spherically symmetric solutions to the Lorentzian constant positive mean curvature flow are stable under sufficiently small initial perturbations, in the sense that the perturbed solutions will always converge spatially-locally and up to a space-time translation to the de Sitter solution.*

I emphasize here that the perturbations are *not* assumed to be spherically symmetric. An immediate corollary is that all *almost* spherically symmetric initial data generating expanding solutions are nonlinearly stable.

To resolve the apparent instability mentioned above requires several technical innovations. First, the poor gauge choice can be ameliorated by replacing the representation of the perturbed solution as a graph over the unperturbed one, by comparing the solutions via their Gauss maps. This "inverse Gauss map" gauge in effect allows us to rewrite the equations of motion as a quasilinear hyperbolic system over a de Sitter space background. This gauge choice can also be regarded as an extension of modulation theory (frequently used in the study of semilinear PDEs [61, 66, 76, 77]) where the modulation parameters are chosen to depend on both space and time (classically the parameters depend only on time). In this sense it combats the formation of spatially localized features due to the cosmological horizons.

Secondly, to properly make use of the inverse Gauss map gauge, from scaling considerations, I replaced the quasilinear wave equation (1) by a first order divergence-curl system derived from the Codazzi equations for the embedding $M \hookrightarrow \mathbb{R}^{1,d}$. This captures the fact that the linear instabilities disappears at the derivative level, another feature of the expanding background space-time. In doing so we need the third innovation, which is the redevelopment of vector field method for such quasilinear first order systems, which involves the replacement of the standard energy-momentum tensor by a Bel-Robinson-like tensor (in this step I drew inspiration from [5] and [80]) and studying t-weighted energy inequalities (similar to those used in the study of cosmological solutions in general relativity [68, 69, 73]).

Future plans

I intend to remove the axial symmetry assumption from Theorem 1. As mentioned above, the main difficulty[2] is that of geodesic trapping. One needs to develop a suitable theory for integrated local energy decay on the background geometry that works for *all* spherical harmonics, while at the same time combat the instability that is already present in the spherically symmetric case. This

[2]From the result of Krieger and Lindblad [48] we see that away from the "throat" where the trapping occurs there are essentially no other obstructions.

is further complicated by the presence of zero-energy resonances for the linear evolution outside axial symmetry (due to the symmetries of the ambient Minkowski space). This likely necessitates the use of modulation theory to control those perturbations.

Another related problem to Theorem 1 is to understand the behavior of the unstable mode in the large. From numerics we can see that that perturbations which shrink the “throat” of the catenoid tend to pinch the throat and lead to finite time singularity formation, while perturbations which expand the throat seem to lead to accelerated expansion with no singularities. Furthermore, the numerics also suggest that the throat pinching singularity is accompanied by the solution degenerating from a time-like hypersurface to a null hypersurface. It would be interesting to verify in one direction the finite time singularity formation and examine the geometric properties of the singular point, and to verify in the other direction the accelerated expansion and find the asymptotic profile.

For the vanishing mean curvature case, another open problem is to reproduce the results of Nguyen and Tian [65] in higher dimensions, namely to demonstrate stable blow-up behavior for initial data being arbitrary compact manifolds, and to resolve the structure of the singularity.

In relation to Theorem 2, there is a technically challenging open problem. In my preprint [81] I also classified all spherically symmetric solutions to the Lorentzian CMC flow with $c = +1$. In addition to the expanding solutions whose general nonlinear stability was settled in [81], there are also a class of solutions which collapse in finite time, as well as a class of intermediate solutions which converge to an Einstein cylinder $\mathbb{R}\times\mathbb{S}^{d-1}$. From the point of view of dynamical systems the Einstein cylinder, which is a stationary solution, represents a hyperbolic fixed point in spherical symmetry. One might ask about the stability properties of the Einstein cylinder without symmetry assumptions. Here we face a major obstacle: since the background space-time is spatially compact and non-expanding, we have no dispersive mechanisms to drive decay of the perturbations. Together with the presence of an unstable mode and several zero-resonances, the analysis of this system appears tricky. On the flip side, the Lorentzian CMC flow equations exhibit extremely good algebraic structure that satisfies multiple types of null conditions. Therefore it is possible that one can demonstrate some sort of finite codimension almost global existence result reminiscent of the work of Delort and Szeftel [19, 20]. Furthermore, the fact that Lindblad [60] was able to prove global existence for the $d = 2$, $c = 0$ equations where dispersive decay is also lacking suggests that there is hope to perhaps even obtain finite codimensional global existence, similar to Theorem 1, for the Einstein cylinder.

II. Anisotropic second-order hyperbolic systems

Many of the techniques used to study small data global existence of quasilinear wave equations, especially the vector field method [42], can be refined for studying systems of nonlinear wave equations with multiple speeds (e.g. [26, 32, 33, 41, 47, 70, 71, 83]). The key shared-feature in these works is that the systems are essentially *isotropic* and hence the linearized operator commutes with the rotation vector fields. This is crucially necessary for the implementation of the vector field method. (In contrast, the classic vector field method uses also the Lorentz boosts; these vector fields can be discarded in favor of the scaling vector field in the multiple-speed case.) Furthermore, that the equations are isotropic also implies that the linearized equations are expected to have the same form of decay rates as one can observe for linear wave equations, since the slowness surfaces are round cones and one can argue (for example) using the usual stationary phase argument.

But generic second-order hyperbolic systems do not have such isotropy, and such systems arise naturally from studying nonlinear elasticity (or more generally Lagrangian theories of maps)

with a pre-stressed background carrying differential stress, and they also arise from crystal optics and crystal acoustics. For these *anisotropic* systems, the classical method of commuting vector fields cannot be directly applied, as the equations do not have rotational symmetry. Furthermore, these systems typically carry intricate null geometry (i.e. the geometry of the slowness surfaces) which can have higher order degeneracies leading to reduced decay rates of the solutions to the linearized equations. (In principle the slowness surfaces of crystal optics and acoustics satisfy quartic and sextic algebraic equations respectively; compare this to the quadratic relations for the wave equation.)

It is an open question how much of the theory concerning the small data global existence and shock formation for quasilinear wave equations (e.g. [1, 2, 8, 15, 42]; also the survey article [28]) can be carried through to the case of anisotropic systems. The state of the art seems to be the results of Liess [53–59]. He derived L^2-L^∞ decay estimates using the Fourier representation of the linearized equations (essentially Van der Corput lemma together with detailed analysis of the geometry of the slowness surface). The reduced decay rates obtained is sufficient to close small data global existence arguments with very high power nonlinearity.

In principle for constant coefficient second order hyperbolic systems the slowness surfaces are locally described by real analytic surfaces of finite type, and so from stationary phase arguments we see that solutions should decay at the rate $t^{-1/k}$ for some $k > 0$ [74]. Furthermore, stationary phase arguments are also highly suggestive of, though insufficient to actually prove, *improved* decay rates for derivatives of the solution tangential to the characteristic hypersurfaces, with an expected gain of t^{-1} for each such derivative. Therefore a suitably strong form of the null condition may in fact suffice to guarantee small data global existence for smooth quasilinear perturbations of such second order hyperbolic systems.

Future plans

As indicated above, the first step would be to develop robust linear decay estimates for anisotropic systems. In particular, one needs results stronger than the stationary phase arguments and in particular capturing improved tangential decay. For the wave equation this is captured through the Klainerman-Sobolev inequalities [43, 45]. To my best knowledge the only existing extension of these inequalities to systems is the work of Chen and Zhou [7] for first order hyperbolic systems under mild assumptions on the principal symbol. Thus I will begin by trying to reproduce the results of [7] in the context of second order regularly hyperbolic systems.

Assuming that the linear decay estimates can be obtained, the goal then would be trying to formulate some generalization of the null conditions such that a dichotomy (similar to that for wave equations [1, 2, 15, 28]) is exhibited. In particular we expect nonlinear systems verifying the null condition to exhibit small data global existence, while systems violating the null condition suffer shock formation.

III. Singularity formation and classification

Given an evolution PDE, we can ask: Do singularities form?

When singularities do form, we want to know: Are they stable? What are the mechanisms driving blow-up? What is the geometric structure of the singularities? Can solutions be continued past the singularities in a suitably weak sense? Is the continuation unique, or does there exist a canonical notion of continuation?

When a PDE arises from a physical model, the formation of singularities usually indicates a breakdown in the utility of the model, or that physically the situation has entered a regime

where the model is no longer applicable. Understanding the mechanisms driving blow-up and the structure of singularities can lead us to understand where the physical model needs to be replaced, and the continuation of solutions past singularities can hint at a deeper physical law being used.

One situation where such a point of view has enjoyed great success is in the study of systems of conservation laws in $1+1$ dimensions [16, 52]. For the hyperbolic conservation laws, it has been well understood that shocks are typical to many models. The understanding of the geometry of the shock front and the methods of continuing solutions led to the discovery that the spaces of functions of bounded variation being the natural space in which to study the weak formation of the problem, and that a canonical continuation of solutions past shock fronts can be given by the physically motivated *entropy conditions*.

Much recent progress has also been made toward understanding the singularity formation for the semilinear wave equation with power nonlinearities

$$\Box u = \pm |u|^{p-1} u. \tag{2}$$

This equation is a model for meson fields in relativistic quantum mechanics [39, 40] and has formal similarities with the equations of equivariant wave maps. In the focusing case where the sign of the nonlinearity is negative, one can classify the singularities into types I and II based on whether the free energy $\|\partial_t u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$ of the solution diverges. In the type II scenario one have many explicit constructions of blow-up solutions [27, 49, 50] as well as detailed information concerning the structure of the singularity [22–24]. Restricted to $1+1$ dimensions, the geometry of the set on which a solution to (2) blows up is also well understood [62–64].

In a joint work with Joachim Krieger [51] we developed a notion of canonical weak extension for type II singular solutions to (2) in 1+3 dimensions with $p = 5$, the critical power. Using the structural results of Duyckaerts, Kenig, and Merle [22, 23], which indicated that type II singularity formation is associated with the bubbling of some number of infinitely rescaled solitons, we considered the time irreversible process of surgically removing those bubbles. This results in a weak solution to (2) for which the energy exhibits a quantized decrease at the blow-up times. Furthermore, we were able to demonstrate that a certain virial-type quantity remains twice continuously differentiable in time for this process, and possesses convexity properties. Combining the two ingredients we find that the canonical weak extension of many known type II blow-up solutions will result, eventually after perhaps some additional type II singularities, in a type I blow-up in finite time.

Two important open problems in PDE theory are also related to understanding singularities and continuations of solutions past them. The first is the goal of finding similar success for hyperbolic conservation laws in higher dimensions, to those discovered in $1+1$ dimensions. Fritz John was able to show that under the assumption of rotational symmetry, where the systems of equation reduce to effectively $1+1$ dimensional PDEs, the aforementioned results carry through with very minor changes [36–38]. But extending the results outside the symmetry class proved to be difficult. One of the obstacles in such extension is that the analysis based on the optimal BV type spaces in $1+1$ dimensions cannot be extended to higher spatial dimensions [67]. The L^2 based existence theorems in higher spatial dimensions unfortunately are less accommodating of the existence of singular points, in particular shock fronts. The state of the art is restricted to the case of small initial data, where the mechanism driving shock formation and the geometry of the shock front are beginning to be understood [1, 2, 15, 28].

A second open problem is the cosmic censorship conjectures in general relativity [14]. Without being precise about the conjecture itself, I note that in symmetry classes (so the resulting equations are effectively $1+1$ dimensional), the conjecture can be resolved [9–13, 17, 18] using, again, BV

type arguments [11]; and the challenge is, again, to extend the result to situations with full spatial degrees of freedom.

Future plans

As already mentioned in section one, there are some related natural questions to the Lorentzian constant mean curvature flow concerning stability of singularity formation and structure of the singular set. In particular, in low dimension Nguyen and Tian [65] were able to extend solutions past the singular set. An interesting question is whether what they observed have analogues in higher dimensions.

Another problem that I will attack is the problem of shock formation for quasilinear systems of wave equations. The shock formation results of Christodoulou [15] can be distilled to a statement about the stability of John's spherically symmetric shock formation mechanism for effectively scalar quasilinear wave equations [28].

Theorem 3 (HKSW 2014)**.** *Given a quasilinear wave equation on $\mathbb{R}^{1,3}$ that fails the null condition. All sufficiently small, non-trivial, compactly supported, spherically symmetric initial data lead to shock formation. Furthermore, for one such shock-forming data, all sufficiently small not-necessarily-spherically-symmetric perturbations will also exhibit shock formation.*

John's spherically symmetric result also extends to quasilinear systems with distinct wave speeds [38]. While we expect Theorem 3 to also hold for systems, to obtain a proof we need to overcome several obstacles, the most severe of which is tied to my interest in anisotropic hyperbolic systems. One of the most important ideas in Christodoulou's work [15] is the observation that, in order to avoid severe derivative loss, one must keep careful track of the true null geometry of the actual solution. This is in stark contrast to the many works in small-data global existence theory where the null geometry used is that of the zero solution, and hence only approximates that of the true solution. Now if we try to extend the results from scalar equations to systems, in the small-data global existence theory, while the true solution may exhibit anisotropy, the background is isotropic and so standard wave equation techniques can apply. But in trying to show shock formation, the true null geometry must be used and we must then contend with issues arising from anisotropy. This is a problem I seek to address.

IV. Geometric methods in hyperbolic PDEs and systems

From the simple observations that the linear wave operator is the Laplace-Beltrami operator for Minkowski space, and that frequency-localized solutions to the wave equation roughly follow light-like geodesics, we see that nonlinear wave equations have strong connections to Lorentzian geometry. Properly used, this connection enables the development of powerful results. In addition to the plethora of results from general relativity where analysis and geometry are tightly coupled, some of the recent progress in PDE theory using geometric methods include Christodoulou's work on shock formation [15]; the low regularity local well-posedness theory of quasilinear wave equations by Klainerman and Rodnianski [44], later also with Szeftel [46], and separately by Wang [75]; and the formulation of the t-conditional Carleman estimates by Ionescu and Klainerman [34, 35]. Proper understanding of the geometry played important roles in many of my works.

In general relativity, in my dissertation [78] I built upon the t-conditional Carleman estimates of Ionescu and Klainerman to demonstrate a uniqueness theorem for charged black holes in general relativity; a particular ingredient needed is the discovery of a characterization of the Kerr-Newman solutions by the vanishing of a certain pair of tensors [79]. This characterization is used again in

a joint work with Pin Yu [82] where we exploited the rigidity of the near-horizon geometry of a stationary black hole to upgrade local geometric restrictions (that a stationary solution is close to the Kerr–Newman solution) to global topological control (that there is only one black hole) through analysis (the construction of essentially a Morse function).

My works on the Lorentzian constant mean curvature flow also showcase how geometry can enhance our understanding of hyperbolic PDEs. For my work on the vanishing mean curvature case [21], of central importance is obtaining a suitable parametrization of the perturbed solution manifold such that the evolution equations exhibit the algebraic structures so as to be tractable. In my work on the positive mean curvature case [81], that I could resolve the apparent linear instability relies entirely on understanding the global causal geometry of the unperturbed solution, and on being able to discover and make use of the inverse Gauss map gauge.

Future plans

In relation to my work on the Lorentzian constant mean curvature flow, and in relation to the recent interest in linear wave equations on curved backgrounds, I will try to understand the trapping phenomenon on direct product manifolds $\mathbb{R} \times \Sigma$ where Σ is Riemannian, asymptotically flat, and exhibits trapped geodesics. The case where Σ is the catenoid is of course directly applicable to understanding Theorem 1 in the general case. But this will also have applications in geometric analysis and general relativity otherwise.

As already alluded to, in order to understand the theory of anisotropic quasilinear hyperbolic systems, a necessary step is to understand their null geometry. Whereas scalar wave equations lead to slowness surfaces defined by quadratic relations, and hence are related to Lorentzian geometry, the slowness surfaces for systems are higher order. Therefore I expect to need to develop a suitable version of Lorentz-Finsler geometry on vector bundles. (While many attempts to define Lorentz-Finsler geometry exist in the literature, none appears suitable for this purpose.) In addition to what I have already outlined in the previous sections, this step will also be crucial for eventually extending the low regularity local wellposedness results for quasilinear wave equations [44, 72, 75] to general systems.

Lastly, recall that the classical Carleman estimates [6, 31] for scalar equations, which are frequently used in control theory and studying unique continuation problems, depend on the notion of pseudo-convexity which is tied to the characteristic structure of the principal symbol. In order to develop a theory that is also applicable to *systems* an understanding of the associated null geometry will be indispensable.

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