On the uniqueness of Kerr-Newman black holes

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Abstract

The uniqueness of the Kerr-Newman family of black hole metrics as stationary asymptotically flat solutions to the Einstein equations coupled to a free Maxwell field is a crucial ingredient in the study of final states of the universe in general relativity. If one imposes the additional requirement that the space-time is axial-symmetric, then said uniqueness was shown by the works of B. Carter, D.C. Robinson, G.L. Bunting, and P.O. Mazur during the 1970s and 80s. In the real-analytic category, the condition of axial symmetry can be removed through S. Hawking's Rigidity Theorem. The necessary construction used in Hawking's proof, however, breaks down in the smooth category as it requires solving an ill-posed hyperbolic partial differential equation. The uniqueness problem of Kerr-Newman metrics in the smooth category is considered here following the program initiated by A. Ionescu and S. Klainerman for uniqueness of the Kerr metrics among solutions to the Einstein vacuum equations. In this work, a space-time, tensorial characterization of the Kerr-Newman solutions is obtained, generalizing an earlier work of M. Mars. The characterization tensors are shown to obey hyperbolic partial differential equations. Using the general Carleman inequality of Ionescu and Klainerman, the uniqueness of Kerr-Newman metrics is proven, conditional on a rigidity assumption on the bifurcate event horizon.

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Contents

Chapter 1

Introduction

The existence of black holes is a fundamental feature of general relativity.[1] To wit, the celebrated Schwarzschild solution was discovered mere months after Einstein published his namesake equations describing celestial evolution. Yet the nature of black holes has proven elusive since the very beginning of relativity theory. When K. Schwarzschild wrote down the first non-trivial exact solution to Einstein's equations in 1916, his goal was that of the gravitational field exterior to a stellar object [35, 34]. It was not until J. Synge [41] and M.D. Kruskal [21] considered the "maximal extension" to the Schwarzschild metric that we began to properly interpret its properties as a black hole. On the other hand, while many open questions still remain in the study of black holes, the explosive growth of the field in the latter half of the twentieth century elucidated features of these special solutions and directed research toward the more pertinent problems.

The importance of black holes was underscored when R. Penrose demonstrated his "singularity theorem"[29, 13]. Prior to that, singular black-hole solutions are sometimes dismissed as unphysical and only manifesting under symmetry assumptions; this is largely because the known black-hole solutions at the time (Schwarzschild,

[1] The historical notes given herein are merely to illustrate and motivate the main problem under discussion. Hence the interpretive history is opinionated, and the descriptive history incomplete; the reader should not take the historical notes to be in any way authoritative.

Reissner-Nordström [31, 27], and Kerr-Newman [20, 26]) are all symmetric exact solutions. In view of Y. Choquet-Bruhat's local-existence theorem for the Cauchy problem in general relativity [11, 5], Penrose's singularity theorem showed the existence of black holes to be "generic" (quite recently D. Christodoulou went further and showed that dynamical formation of black holes is also generic [6]). The study of black holes gained further prominence through investigations of the long-time-existence aspect of the Cauchy, or initial value, problem. Einstein's equations are known to be reducible to a system of quasilinear wave equations (it is in this formulation that Choquet-Bruhat proved local existence), and hence some wave-like or dispersive phenomena are expected[2]. Precisely this was shown by H. Bondi et al.[2] and R. Sachs [33] via a mass-loss formula that described energy being carried away from a local source through gravitational radiation. For dispersive or wave-type systems, it seems reasonable to expect that a solution consists of two parts: a localized stationary part where attractive nonlinearities (e.g. gravity) overcome the dispersive tendencies, and a radiative part which "decays" over time (the archetypal example for this splitting being the Korteweg-de Vries equation; see Chapter 4 in [42] and references therein). Whether such a characterization (a property technically termed "scattering") actually holds is the subject of active research, and considerations of the "stationary part" brings us back to the subject of black holes.

As mentioned above, the explicit closed-form black-hole solutions are all highly symmetrical. In fact, the Kerr-Newman family of solutions (which subsumes the Schwarzschild and Reissner-Nordström metrics) have a time symmetry that qualifies them as stationary (see Section 1.3 for definitions). Thus they are candidates for the possible final state of evolution for a given space-time. The natural question to ask then, is

Problem 1.0.1. *Does the Kerr-Newman family constitute the only candidates for the*

[2]The subject of gravitational waves was (and perhaps still is) a contentious one. The reader is referred to the excellent book by D. Kennefick [19] for a more complete history.

possible final states?

The present work is an effort to address the above problem. Written as such, of course, the question is not well-posed mathematically. In the remainder of this chapter, some technical definitions will be made and, in Section 1.4, a more precise statement of the problem under consideration will be given.

1.1 The Einstein-Maxwell equations

First it is necessary to describe the physical system: what is a space-time, what sorts of matter are considered, and what are the physical laws of evolution?

Definition 1.1.1. *A* space-time *shall refer to a pair* $(\mathcal{M}, g_{ab})$ *such that:*

- $\mathcal{M}$ *is a four-dimensional, paracompact, orientable, smooth manifold.*
- g_{ab} *is a smooth Lorentzian metric on* $\mathcal{M}$*. In other words,* g_{ab} *is a smooth* $(0,2)$*-tensor field, is symmetric and non-degenerate, and has signature* $(-,+,+,+)$*.* g^{ab} *stands for the metric inverse. All index-raising and -lowering will be with respect to* g^{ab} *or* g_{ab} *as appropriate.*
- $\mathcal{M}$ *is time-orientable relative to the metric* g_{ab} *(in other words, there exists a continuous, globally non-vanishing vector field* T_0 *such that* $g_{ab}T_0^aT_0^b < 0$*).*

The only allowed matter field is a Maxwell field which describes electro-magnetism.

Definition 1.1.2. *A* Maxwell field *or* Maxwell two-form *shall refer to a real-valued, smooth two-form* H_{ab} *on* $\mathcal{M}$*, such that Maxwell's equations are satisfied:*

$$\nabla_{[a}H_{bc]} = 0$$

$$\nabla^a H_{ba} = 0$$

where ∇_a is the Levi-Civita connection for the metric g_{ab}, brackets $[\cdot]$ around indices denote full anti-symmetrization, and Einstein's convention of contracting repeated indices is in force.

Definition 1.1.3. *The triple $(\mathcal{M}, g_{ab}, H_{ab})$ is said to be a* solution *to the* Einstein-Maxwell*, or* electro-vac*, system if H_{ab} is a Maxwell field and Einstein's equations are satisfied:*

$$R_{ab} - \frac{1}{2} R g_{ab} = T_{ab}$$

where R_{ab} is the Ricci curvature of the metric g_{ab}, $R = g^{ab} R_{ab}$ is the scalar curvature, and $T_{ab} = 2H_{ac}H_b{}^c - \frac{1}{2} g_{ab} H_{cd} H^{cd}$ is the rescaled stress-energy tensor for the Maxwell field.

By construction, the stress-energy tensor of the Maxwell field is trace-free $g_{ab}T^{ab} = 0$. By taking the trace of Einstein's equations, the scalar curvature vanishes. Thus a solution to the Einstein-Maxwell system must have $R = 0$ and $R_{ab} = T_{ab}$.

1.2 Causal geometry

The Lorentzian metric g_{ab} imposes a structure on the tangent space $T_p\mathcal{M}$ at any point $p \in \mathcal{M}$. With a suitable choice of basis vectors, $T_p\mathcal{M}$ can be identified with the Minkowski space, whence it is possible to decompose $T_p\mathcal{M}$ into the disjoint union of ${}^tT_p\mathcal{M} \cup {}^sT_p\mathcal{M} \cup {}^nT_p\mathcal{M}$, where elements of

$$ {}^tT_p\mathcal{M} := \{v^a \in T_p\mathcal{M} | g_{ab} v^a v^b < 0\} \tag{1.1a}$$

are said to be time-like, elements of

$$ {}^sT_p\mathcal{M} := \{v^a \in T_p\mathcal{M} | g_{ab} v^a v^b > 0\} \tag{1.1b}$$

are said to be space-like, and elements of

$$ {}^{n}T_p\mathcal{M} := \{v^a \in T_p\mathcal{M} | g_{ab}v^a v^b = 0\} \tag{1.1c} $$

are said to be light-like or null.

It is clear from the Minkowski-space picture that ${}^{n}T_p\mathcal{M}$ is a double-cone, and that ${}^{t}T_p\mathcal{M}$ has two connected components. By the assumption that $(\mathcal{M}, g_{ab})$ is time-orientable, there exists a continuous choice

$$ {}^{t\pm}T_p\mathcal{M} := \{v^a \in {}^{t}T_p\mathcal{M} | \pm g_{ab}v^a T_0^b < 0\} \tag{1.2} $$

where ${}^{t+}T_p\mathcal{M}$ consists of the future-pointing time-like vectors, and ${}^{t-}T_p\mathcal{M}$ of the past-pointing time-like vectors. Similarly ${}^{n}T_p\mathcal{M} \setminus \{0\}$ can also be decomposed into ${}^{n\pm}T_p\mathcal{M}$.

A C^1 curve γ in $\mathcal{M}$ is said to be time-like (similarly space-like or null) if its tangent vector at every point is time-like. The curve is said to be causal if its tangent vector at every point is either time-like or null. Notice that for a given parametrization of a causal C^1 curve γ, its time orientation is fixed, and reversing the parametrization gives a reversed time orientation.

Now, given two points p, q in $\mathcal{M}$, p is said to be to the future of q, and written $p \succcurlyeq q$ if there exists a future pointing causal C^1 curve $\gamma : [0, 1] \to \mathcal{M}$ with $\gamma(0) = q$ and $\gamma(1) = p$; the strict inequality $p \succ q$ is taken when γ is strictly time-like. (For properties of these causal relations, see Chapter 14 in [28].) Let A be a non-empty subset of $\mathcal{M}$, its future set is defined to be

$$ I^+(A) = \{p \in \mathcal{M} | \exists q \in A : q \prec p\} \ , $$

and its causal future set is

$$J^+(A) = \{p \in \mathcal{M} | \exists q \in A : q \preccurlyeq p\} \ .$$

The past sets I^- and J^- can be defined analogously. The notation $I(A, B)$ (similarly $J(A, B)$) is used to mean $I^+(A) \cap I^-(B)$.

To rule out pathological examples, the following condition is often applied to space-times

Definition 1.2.1. *The space-time $(\mathcal{M}, g_{ab})$ is said to be* strongly causal *if given any point $p \in \mathcal{M}$ and a neighborhood U of p, it is possible to find a neighborhood $p \in V \subset U$ such that every causal curve with endpoints in V lie entirely in U.*

A stronger condition is

Definition 1.2.2. *An open subset $Q \subset \mathcal{M}$ is said to be* globally hyperbolic *if it is strongly causal and that for any two points $p, q \in Q$, the set $J(p, q)$ is compact.*

The following definition is standard

Definition 1.2.3. *A subset Σ of $\mathcal{M}$ is said to be a* Cauchy hypersurface *if every inextendible time-like curve meets Σ exactly once.*

It is well-known (see Lemma 14.29 in [28]) that a Cauchy hypersurface is a closed achronal topological hypersurface and is met by every inextendible causal curve exactly once. It is also well-known (see Corollary 14.39 in [28]) that a sufficient condition for a space-time to be globally hyperbolic is for it to have a Cauchy hypersurface.

1.3 Stationary black hole solutions

The classical definition of a general black hole (see [13], Chapter 9 for an example) depends on the regularity concepts of *asymptotic predictability* (and associated *asymptotic simplicity*). Since this work focuses only on stationary space-times, we will forgo the customary definition of black holes from null infinities, and instead use a definition more adapted to the work at hand [12] (see also [8]).

A space-time $(\mathcal{M}, g_{ab})$ is said to be *stationary asymptotically flat* if it admits a one parameter group of isometries Φ_t and contains what is called an *asymptotic end* $\mathcal{M}^\infty$ on which the generators of Φ_t are uniformly time-like. $\mathcal{M}^\infty$ is required to be an open submanifold of $\mathcal{M}$ such that $\mathcal{M}^\infty = \cup_{t\in\mathbb{R}}\Phi_t\Sigma$ where Σ is a space-like hypersurface diffeomorphic to $\mathbb{R}^3$ minus a ball. In the coördinates $t \times \mathbb{R}^3$ thus induced, g_{ab} satisfies certain decay conditions as one approaches infinity on $\mathbb{R}^3$ (which is automatically uniform in t by the isometry assumption; the decay conditions will be made more explicit in Chapters 3 and 4). The decay condition essentially states that the metric approaches that of the Minkowski metric near infinity; the exact rate of decay is not important in this section. For a more precise definition of stationary asymptotically flat, see Definition 2.1 for "(k, α)-asymptotically stationary" in [8]. In the Einstein-Maxwell case, we will also require that the Maxwell field H_{ab} inherits the symmetry (i.e. $\Phi_{t*}H_{ab} = H_{ab}$)[3], and that it satisfies suitable decay conditions.

Assuming the space-time is globally hyperbolic, we consider the following sets $\mathfrak{B} = \mathcal{M} \setminus I^-(\mathcal{M}^\infty)$ and $\mathfrak{W} = \mathcal{M} \setminus I^+(\mathcal{M}^\infty)$. $\mathfrak{B}$ and $\mathfrak{W}$ are called the "black hole" and "white hole" regions relative to the end $\mathcal{M}^\infty$. We also write $\mathfrak{D} = \mathcal{M} \setminus (\mathfrak{B} \cup \mathfrak{W})$ for the domain of outer communication. The future and past event horizons $\mathfrak{H}^\pm$ are defined to be the boundaries of $\mathfrak{B}$ and $\mathfrak{W}$ respectively. Since the space-time is globally hyperbolic, $\mathfrak{H}^\pm$ are achronal sets (any two points cannot be connected by a time-like curve) generated by null geodesic segments.

The interpretation of the above definitions is that the black/white hole regions have interesting causal relations to the exterior region. Since the causal future $J^+(\mathfrak{B})$ of the black hole is disjoint from $\mathcal{M}^\infty$, it is impossible to send a signal from inside

[3] Unlike in the Einstein-scalar-field case, this condition is not automatically satisfied. Some analysis to non-inheriting case was performed by Tod [44].

the black hole to an observer outside the black hole. Similarly, it is impossible for an observer inside the white hole to receive signals from outside the white hole. In other words, a black hole region is one from which light cannot escape while the white hole region is one into which light cannot penetrate.

The most well-known family of exact, closed-form, black-hole solutions is probably the Kerr-Newman family. In Boyer-Lindquist coördinates, the Kerr-Newman line element ds^2 and the associated vector potential A for the Maxwell field are given as

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr - q^2}{r^2 + a^2\cos^2\theta}\right) dt^2 - \frac{2a(2Mr - q^2)}{r^2 + a^2\cos^2\theta}\sin^2\theta d\phi dt \\ &+ \sin^2\theta\left(r^2 + a^2 + \frac{a^2(2Mr - q^2)}{r^2 + a^2\cos^2\theta}\sin^2\theta\right) d\phi^2 \\ &+ \frac{r^a + a^2\cos^2\theta}{r^2 + a^2 - 2Mr + q^2} dr^2 + (r^2 + a^2\cos^2\theta) d\theta^2 \ , \end{aligned} \tag{1.3}$$

$$A = -\frac{qr}{r^2 + a^2\cos^2\theta} dt + \frac{qra\sin^2\theta}{r^2 + a^2\cos^2\theta} d\phi \ ; \tag{1.4}$$

it represents a charged, spinning black hole. If we set the charge parameter $q = 0$, we reduce to the Kerr subfamily. If we set the angular momentum parameter $a = 0$, we reduce to the Reissner-Nordström family. And if both charge and angular momentum vanishes, the black hole described is Schwarzschild. The free parameter M represents the mass of the black hole, and it is generally assumed, based on physical interpretations, that $a^2 + q^2 \leq M^2$; in the case of equality the black hole is said to be "extremal". If we set $r = M + \sqrt{M^2 - a^2 - q^2}$, we see that the metric becomes singular in these coördinates: this is a coördinate singularity (not a physical singularity) representing the event horizon of the metric.

The properties of stationary black holes, especially those of their event horizons, are well-studied. The topological uniqueness of black holes (see, for example, [7]) guarantees that for a stationary asymptotically flat black hole solution to the Einstein-Maxwell equations, the domain $\mathfrak{D}$ is simply connected, and each connected component of $\mathfrak{H}^\pm$ must have topology $\mathbb{S}^2 \times \mathbb{R}$ (in this work we shall assume that there is only one connected component of the horizon). Furthermore, from the definitions above, it is clear that $\mathfrak{H}^\pm$ are invariant under the flow Φ_t. This immediately implies that the associated Killing vector field must be tangent to the event horizon. If we further assume that the only fixed points of $\Phi_t|\mathfrak{H}^\pm$ live on $\mathfrak{H}_0 := \mathfrak{H}^+ \cap \mathfrak{H}^-$, then the Killing vector field must be either null or space-like on $\mathfrak{H}^\pm$. In the case the Killing vector field is null on the horizon, Sudarsky and Wald [40] showed that the space-time must be static, and hence Reissner-Nordström (see the next section). In this work we will only consider the case when the Killing vector field is space-like somewhere on the horizon. We also make the assumption that the future and past event horizons $\mathfrak{H}^\pm$ are smooth null hypersurfaces that intersect transversely at $\mathfrak{H}_0$. This assumption is related to the non-degeneracy of the event horizon [4] which is associated to the non-vanishing of surface gravity [45]. Physically this assumption may be justified by the expectation that degenerate event horizons correspond to extremal black holes (those for which the sum of the normalized angular momentum and charge equals the mass), which are thought to be unphysical. Another property of stationary black holes is Hawking's area theorem [13], which tells us that the null mean curvature for $\mathfrak{H}^\pm$ vanishes (for some consequences of this see Section 2.5). The precise formulation of the assumptions mentioned here will be given in Chapter 4.

1.4 The problem of uniqueness

A large open problem in the classical study of black holes is the *Final State Conjecture*, which contains as part of it Problem 1.0.1 mentioned above. The conjecture is extremely open, in the sense that even a reasonable formulation of the conjecture is unknown. Roughly speaking, one way to state the conjecture is

Conjecture 1.4.1. *[Final State] For a generic asymptotically flat, globally hyper-*

bolic solution to the Einstein-Maxwell equation (possibly coupled with other fields), we can find a foliation Σ_t such that the solution, when restricted to Σ_t, converges to a superposition of multiple Kerr-Newman black holes as $t \to \infty$.

It is entirely unknown what "generic" means, or whether actually other fields are allowed, or even how to pick a foliation (whether the foliation is by a time-function or by asymptotically hyperbolic slices or some more exotic construction), or in what sense can we take the convergence. As part of the effort to better understand the conjecture, two natural, possibly easier, problems are asked. One is Problem 1.0.1 above; the other is the problem of nonlinear stability of black holes

Problem 1.4.2. *Considering the Cauchy problem in general relativity. If one prescribes an initial data set (see [45, 13] for a description of the Cauchy problem) that is close to a Kerr-Newman black hole, will the evolution converge toward a possibly different Kerr-Newman black hole?*

As described in the opening paragraphs, the existence of gravitational waves seems to lend a mechanism for radiative decay of solutions, and hence the expectation is that the nonlinear stability problem will be answered in the affirmative sometime in the future.

Let us now focus on the problem of uniqueness. From a physical perspective, it is natural to expect that candidates for the final states are stationary solutions. Unfortunately, unlike classical evolution equations, Einstein's equation does not admit a simple globally, canonically, defined time. One cannot simply prescribe

$$\frac{\partial}{\partial t}\text{System} = 0$$

and solve an elliptic system. A more geometric prescription would be to "find a solution to the Einstein-Maxwell system that admits a globally time-like Killing vector field." In view of known closed-form exact solutions, however, this prescription is overly restrictive, as in the Kerr-Newman family there exists the *ergoregion*, which is contained in the physical region $\mathfrak{D}$, where the global Killing vector field becomes space-like. (Observe that in the Boyer-Lindquist coördinates (1.3), the ergoregion is defined by

$$M + \sqrt{M^2 - a^2 \cos^2\theta - q^2} > r > M + \sqrt{M^2 - a^2 - q^2}$$

and pictorially is an oblate spheroidal region surrounding the event horizon.) Hence we are forced to adopt the formulation for stationary asymptotically flat as described in the previous section. That the symmetry is only time-like at a neighborhood of infinity implies that reducing the equations by this symmetry will not lead to a elliptic system, and hence such a naïve argument will not yield uniqueness of solutions.

Symmetry assumptions, however, can be useful in establishing uniqueness of solutions. As mentioned before, the closed-form metrics of the Schwarzschild, Reissner-Nordström, Kerr, and Kerr-Newman families are all highly symmetrical. In particular, the former two are static (possess a stationary Killing vector field that is hypersurface-orthogonal; cf. Frobenius' theorem) and spherically symmetric (admit an action of $SO(3)$ whose orbits are space-like); the latter two are stationary and axially symmetric (admit an action of $U(1)$ with space-like orbits, which commutes with the stationary symmetry). Historically the first non-trivial uniqueness result in general relativity is Birkhoff's theorem (which was known since the 1920s [18]), which states that any spherically symmetric solution to the Einstein vacuum equations must be the Schwarzschild solution. In particular, spherical symmetry is enough to imply staticity. This result is later generalized to the electrovac system for the Reissner-Nordström solutions. Birkhoff's theorem is possible because the spherical symmetry reduces Einstein's equation to a problem in $1 + 1$ dimensions, where the Lorentzian geometry automatically imposes many constraints to reduce the problem to essentially ordinary differential equations which can be directly integrated. The next step forward came in the 1960s, when W. Israel established [16, 17] what is, loosely speaking, the converse of Birkhoff's theorem: a static, asymptotically flat space-time that is regular on the event horizon must be spherically symmetric. Israel's theorem exploited the fact that a static space-time does not admit an ergoregion (this roughly follows via a maximum principle on the Lorentzian norm of the static Killing vector field: it vanishes on the black hole boundary and is harmonic where it does not vanish, so it must be time-like in all of the exterior region), and thus Einstein's equation reduces to a degenerate elliptic system (degenerate near the horizon where the Killing vector field becomes null) for which uniqueness can be shown. B. Carter's 1973 Les Houches report [4] finally sparked an attempt to similarly characterize the Kerr and Kerr-Newman families: he showed that asymptotically flat, stationary, and axially-symmetric solutions to the vacuum (electrovac) equations form a two-parameter (three-) family. Between D.C. Robinson [32], P.O. Mazur [24], and G.L. Bunting [3], Carter's program was completed and the Kerr and Kerr-Newman families are established as essentially the unique solutions to the asymptotically flat, stationary, axially-symmetric Einstein's equations. As explicated by Bunting's work, the assumption of axial symmetry is essential in Carter's program: while the stationary Killing vector field is no longer time-like everywhere in $\mathfrak{D}$, the span of the stationary Killing vector field *and* the axial symmetry is Lorentzian. In other words, there is a Riemannian structure on the space of orbits under the Abelian symmetry group generated by the stationary isometry and the axial symmetry. Under this identification, the Einstein-Maxwell system is reduced to a harmonic map with singular boundary conditions, for which uniqueness follows from elliptic theory.

At this point, the powerful results about the event horizons of stationary black holes came into play. Hawking, starting from the area theorem, deduced that on the event horizon of an arbitrary stationary black hole there must exists an axial symmetry [13]. If one is allowed to then assume that the space-time is real-analytic, one can in principle "solve" the Killing equation to extend the axial symmetry to the entire space-time. By appealing to the Carter-Robinson-Mazur-Bunting theorem, he is able to conclude the famous No Hair Theorem, that any real-analytic nondegenerate stationary asymptotically flat solution to the Einstein-Maxwell system with a connected event horizon must be a non-extremal Kerr-Newman black hole.

Hawking's result, however, strongly depends on the analyticity of the space-time, which is not completely a priori available. Indeed, in view of the work of H. Müller zum Hagen [25] and P. Tod [43], we can expect that in the portion of $\mathfrak{D}$ where the stationary Killing vector field is strictly time-like, a solution to the Einstein-Maxwell system is analytic in harmonic coördinates. The argument, which depends on elliptic regularity, breaks down inside the ergoregion. In view of this, it is desirable to be able to obtain a No Hair Theorem without the analyticity assumption. To address this issue, A. Ionescu and S. Klainerman [14, 15] initiated a program to study the uniqueness problem in the smooth category.

Problem 1.4.3. *Consider a smooth solution of the Einstein-Maxwell equations. Assume the space-time is stationary asymptotically flat and globally hyperbolic, and that the event horizon is connected and nondegenerate. Can we say that the space-time must be isometric to a Kerr-Newman solution?*

In the vacuum case, Ionescu and Klainerman gave a conditional answer in the affirmative to the above problem. The principal argument is as follows: the bifurcate event horizon is a characteristic hypersurface for the wave operator. While for the exterior problem, the characteristic initial value problem is ill-posed, often one can obtain uniqueness of solutions should they exist. The Einstein-Maxwell system can be written as tensorial wave equations for the curvature tensor and the Maxwell two-form. The hope then is that for a given initial data on the bifurcate event horizon, one can use Carleman type estimates to show that there can be at most one solution with those data. Then if one can show that the initial data corresponding to Kerr-Newman space-time is the only reasonable initial data (which one can heuristically hope for because Hawking's result *on* the bifurcate event horizon can still hold without using the analyticity assumption). This last step, however, cannot be completed. Instead, one can show that the results arising from the study of the event horizon allows one to reduce the uniqueness of initial data to scalar conditions on the bifurcate sphere, which while we hope can be removed, is currently necessarily prescribed as an assumption. For the case when the Maxwell field is assumed to vanish identically, Ionescu and Klainerman carried out the above program. In other words, Ionescu and Klainerman's result showed that, in the vacuum case, Problem 1.4.3 can be reduced to asking whether the bifurcate sphere of a stationary asymptotically flat solution must agree with the bifurcate sphere of a Kerr space-time.

In this work, we extend Ionescu and Klainerman's result to cover the Einstein-Maxwell case. In particular, we show the following

Theorem 1.4.4. *Consider a smooth solution of the Einstein-Maxwell equation; assume the solution is stationary asymptotically flat and globally hyperbolic, and that the event horizon is connected and nondegenerate. Furthermore assume that the bifurcate sphere $\mathfrak{H}_0$ of the solution satisfied some rigidity assumptions that is known to be satisfied by a Kerr-Newman space-time, then the domain of outer communication $\mathfrak{D}$ of the solution is everywhere locally isometric to a Kerr-Newman space-time.*

The precise conditions for the theorem and the rigidity assumption on the bifurcate sphere will be laid out in Chapter 4.

1.5 Organization and overview of the present work

In Chapter 2, we review some well-known and some not-so-well-known facts about Lorentzian geometry in four-dimensions in the presence of a stationary Killing vector field. In particular, we recall the notion of anti-self-dual two-forms and anti-self-dual Weyl fields, upon which language the main hypotheses of the various theorems in this work is stated. We also recall the tetrad formalism of Ionescu-Klainerman, which is a computational aid similar to the Newman-Penrose formalism, but slightly better adapted to the natural symmetry of swapping the two principal null vectors in a space-time with Petrov type D. In the sequel, the anti-self-dual forms and the tetrad formalism consist the main tools for calculating *tensor* (or *invariant*) and *scalar* (or *frame-dependent*) expressions respectively.

In Chapter 3, we extend a result of Mars [22] to the electrovac case. Mars obtained a tensorial characterization of the Kerr space-time. As a starting point, consider the Minkowski space as a solution to the Einstein vacuum equations. By the vacuum equations any solution is automatically Ricci flat. To characterize Minkowski space, it therefore suffices to require the Weyl conformal tensor to vanish. Furthermore, such a characterization is local: given a solution of the Einstein vacuum equations, supposing that the Weyl conformal tensor vanishes on an open set $\mathcal{U}$, then we can conclude that $\mathcal{U}$ is locally isometric to a subset of Minkowski space. Similarly, to characterize Kerr space-time, Mars showed that it suffices to ask for an algebraically-Weyl field (by which we mean a $(0,4)$ tensor field that is trace-free in all pairs of indices, that is antisymmetric in the first two and the last two, and is symmetric when considered as a map from two-forms to two-forms) to vanish. The important feature is that this algebraically-Weyl field can be constructed invariantly using the conformal Weyl curvature and the stationary Killing vector field. Therefore one obtains a tensorial characterization of Kerr space-times among all stationary solutions to the Einstein vacuum equations. Furthermore, this characterization is essentially local [23]. In the work of Ionescu-Klainerman, it is by showing that this characterization tensor vanishes identically that they show the space-time is locally isometric to Kerr.

In this work, a tensorial characterization for Kerr-Newman space-time among stationary solutions to the Einstein-Maxwell equations is constructed. Because of the inclusion of the matter field, it is necessary that the characterization uses two tensors: an algebraically-Weyl one to control "gravitational waves" as in the Kerr case, and a two-form to control the "electromagnetic waves" which is the new feature of this characterization. The two tensors are shown to be invariantly constructed from the Weyl curvature tensor, the Maxwell field of the solution, and the stationary Killing vector field. And the characterization is again, essentially local (see Theorem 3.2.1).

In Chapter 4, we run through the same argument as in Ionescu-Klainerman [14]. First we show that the characterization tensors for the Kerr-Newman space-time obey nonlinear wave equations through a long and tedious computation. Next we show that the two tensors can be made to vanish on the bifurcate event horizon provided certain scalar conditions on the bifurcate sphere is satisfied. An application of Ionescu and Klainerman's generalized Carleman inequality then shows that the two tensors must vanish in the domain of outer communication. To apply the Carleman estimate, however, one needs a set of conditional pseudo-convex weights. And for this it is crucial that the characterization of Kerr-Newman space-time is essentially local: we can use a bootstrapping procedure to control the pseudo-convex weights. Once a neighborhood is shown to have vanishing characterization tensor, the local isometry allows us to get a better control on the pseudo-convex weights than we assumed. The standard method of continuity then allows us to conclude that the set on which the characterization tensor vanishes is both open and closed in $\mathfrak{D}$, and hence is the entire domain of outer communication.

Much of the material in Chapters 2 and 3 have been accepted for publication in Annales Henri Poincaré. Most of the material in Chapter 2 are previously known, though the presentation may be different; the main exception is Section 2.4 which generalizes the Ionescu-Klainerman tetrad formalism to include Ricci terms. Theorem 3.5.1 is a recent addition, and has not appeared in print prior. Chapter 4 is entirely new, insofar as any generalization of a previously known result can be.

1.6 Notational conventions

Lastly, we define the following notational shorthand for Lorentzian "norms" of tensor fields. For an arbitrary (j,k)-tensor $Z^{a_1 a_2 \ldots a_j}_{b_1 b_2 \ldots b_k}$, we write

$$Z^2 = g_{a_1 a_1'} g_{a_2 a_2'} \cdots g_{a_j a_j'} g^{b_1 b_1'} \cdots g_{b_k b_k'} Z^{a_1 a_2 \ldots a_j}_{b_1 b_2 \ldots b_k} Z^{a_1' a_2' \ldots a_j'}_{b_1' b_2' \ldots b_k'}$$

for the inner-product of $Z^{\cdots}_{\cdots}$ with itself. Note that in the semi-Riemannian setting, Z^2 can take arbitrary sign.

Chapter 2

Geometric background

In this chapter, some previously-known geometrical results are summarized, and the results of some calculations (which will be used in the sequel) are recorded.

2.1 Anti-self-dual two forms and curvature decomposition

As is well known, the Riemann curvature tensor R_{abcd} for an n-dimensional semi-Riemannian manifold admits a decomposition

$$R_{abcd} = W_{abcd} + \frac{1}{n-2}(g \owedge S)_{abcd} + \frac{R}{2n(n-1)}(g \owedge g)_{abcd} \ , \tag{2.1}$$

where $S_{ab} = R_{ab} - \frac{1}{n}Rg_{ab}$ is the traceless Ricci tensor, and $\owedge$ is the Kulkarni-Nomizu product taking two $(0,2)$-tensors to a $(0,4)$-tensor

$$(h \owedge k)_{abcd} := h_{ac}k_{bd} + h_{bd}k_{ac} - h_{ad}k_{bc} - h_{bc}k_{ad} \ . \tag{2.2}$$

Notice that the Kulkarni-Nomizu product of two symmetric $(0,2)$-tensors automatically satisfies all algebraic properties of the Riemann curvature tensor.

This decomposition is fundamentally related to the invariants of the curvature tensor under (indefinite) orthogonal rotations in the tangent space; in four dimensions it is also crucially related to self-dual and anti-self-dual two forms. These two facts were first clarified by I.M. Singer and J.A. Thorpe in the Riemannian case [38], though their results also can be adapted to the semi-Riemannian situation with minimal changes [1]. Here two of their main results are reproduced.

Theorem 2.1.1 (Singer-Thorpe '69 A)**.** *Let* (V, g_{ab}) *be a scalar product space (*g_{ab} *is a symmetric, non-degenerate, bilinear form on* V*), and* Λ^2 *the space of (antisymmetric) two-vectors. Let* R_{abcd} *be a (0,4)-tensor corresponding (via* g_{ab}*) to a symmetric map* $\Lambda^2 \to \Lambda^2$*. Then there exists a decomposition*

$$R_{abcd} = R^{(1)}_{abcd} + R^{(2)}_{abcd} + R^{(3)}_{abcd} + R^{(4)}_{abcd} \ ,$$

where the $R^{(i)}_{abcd}$ *are mutually orthogonal under the norm* $\langle R, S\rangle = R_{abcd}S^{abcd}$ *where index-raising is relative to* g_{ab}*. Furthermore,* $R^{(1)}_{abcd}$ *is the only one not satisfying the first Bianchi identity,* $R^{(2)}_{abcd}$ *is the only one with non-zero scalar curvature (the double trace* $R^{(i)}_{abcd}g^{ac}g^{bd} = 0$ *if* $i \neq 2$*), and* $R^{(3)}_{abcd}$ *is the only one with non-vanishing traceless Ricci part.*

In particular, for a Riemann curvature tensor, $R^{(1)}_{abcd} = 0$, $R^{(2)}_{abcd} = \frac{R}{2n(n-1)}(g \owedge g)_{abcd}$, $R^{(3)}_{abcd} = \frac{1}{n-2}(g \owedge S)_{abcd}$, and $R^{(4)}_{abcd} = W_{abcd}$.

In four-dimensional setting, the Hodge star operator $*$ also is a symmetric map from Λ^2 to itself. The decomposition in the above theorem satisfies the following,

Theorem 2.1.2 (Singer-Thorpe '69 B)**.** *The linear maps given by* $R^{(i)}_{abcd}$ *are characterized by:*

[1] The differences introduced by a semi-Riemannian setting can be solved with, for instance, Lemma 3.40 in [28]; see the proof of Proposition 3.41 *ibid* (and compare to the proof in the Riemannian case) for an illustration.

- $R^{(1)}_{abcd}$ *is a multiple of the volume form (in other words, as a linear map from* Λ^2 *to itself, it is a multiple of the Hodge star* $*$*);*

- $R^{(2)}_{abcd}$ *is a multiple of identity map from* Λ^2 *to itself. In coördinates this means it is a multiple of* $(g \owedge g)_{abcd}$.

- $R^{(3)}_{abcd}$ *anti-commutes with* $*$*;*

- $R^{(4)}_{abcd}$ *commutes with* $*$*, and its trace and the trace of its composition with* $*$ *both vanishes.*

2.1.1 Complex anti-self-dual two forms

On a four dimensional Lorentzian space-time $(\mathcal{M}, g_{ab})$, the Hodge-star operator $* : \Lambda^2 T^*\mathcal{M} \to \Lambda^2 T^*\mathcal{M}$ is a linear transformation on the space of two-forms. In index notation,

$$^*X_{ab} = \frac{1}{2}\epsilon_{abcd}X^{cd} ,$$

where ϵ_{abcd} is the volume form and index-raising is done relative to the metric g. Since the metric signature is $(-,+,+,+)$, the double dual is seen to be $** = -Id$, which introduces a complex structure on the space $\Lambda^2 T^*\mathcal{M}$. By complexifying and extending the action of $*$ by linearity, $\Lambda^2 T^*\mathcal{M} \otimes_{\mathbb{R}} \mathbb{C}$ can be split into the eigenspaces $\Lambda_\pm$ of $*$ with eigenvalues $\pm i$. An element of $\Lambda^2 T^*\mathcal{M} \otimes_{\mathbb{R}} \mathbb{C}$ is said to be *self-dual* if it is an eigenvector of $*$ with eigenvalue i, and *anti-self-dual* if it has eigenvalue $-i$. It is easy to check that given a real-valued two-form X_{ab}, the two-form

$$\mathcal{X}_{ab} := \frac{1}{2}(X_{ab} + i^*X_{ab}) \tag{2.3}$$

is anti-self-dual, while its complex conjugate $\bar{\mathcal{X}}_{ab}$ is self-dual.

In the sequel, elements of $\Lambda^2 T^*\mathcal{M}$ shall be written with upper-case Roman letters, and their corresponding anti-self-dual forms with upper-case calligraphic letters. The projection

$$X_{ab} = \mathcal{X}_{ab} + \bar{\mathcal{X}}_{ab}$$

is a natural consequence of (2.3).

Following are some product properties [22] of two-forms:

$$X_{ac}Y_b{}^c - {}^*X_{ac}{}^*Y_b{}^c = \frac{1}{2}g_{ab}X_{cd}Y^{cd} \ , \tag{2.4a}$$

$$X_{ac}{}^*X_b{}^c = \frac{1}{4}g_{ab}X_{cd}{}^*X^{cd} \ , \tag{2.4b}$$

$$\mathcal{X}_{ac}\mathcal{Y}_b{}^c + \mathcal{Y}_{ac}\mathcal{X}_b{}^c = \frac{1}{2}g_{ab}\mathcal{X}_{cd}\mathcal{Y}^{cd} \ , \tag{2.4c}$$

$$\mathcal{X}_{ac}\mathcal{X}_b{}^c = \frac{1}{4}g_{ab}\mathcal{X}_{cd}\mathcal{X}^{cd} \ , \tag{2.4d}$$

$$\mathcal{X}_{ac}X_b{}^c - \mathcal{X}_{bc}X_a{}^c = 0 \ , \tag{2.4e}$$

$$\mathcal{X}_{ab}Y^{ab} = \mathcal{X}_{ab}\mathcal{Y}^{ab} \ , \tag{2.4f}$$

$$\mathcal{X}_{ab}\bar{\mathcal{Y}}^{ab} = 0 \ . \tag{2.4g}$$

Now, the projection operator $\mathcal{P}_\pm : \Lambda^2 T^*\mathcal{M} \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda_\pm$ can be given in index notation as

$$(\mathcal{P}_+X)_{ab} = \bar{\mathcal{I}}_{abcd}X^{cd} \ ,$$
$$(\mathcal{P}_-X)_{ab} = \mathcal{I}_{abcd}X^{cd} \ ,$$
$$\text{where } \mathcal{I}_{abcd} = \frac{1}{4}(g_{ac}g_{bd} - g_{ad}g_{bc} + i\epsilon_{abcd}) \ .$$

With the complex tensor $\mathcal{I}_{abcd}$, it is possible to define

$$(\mathcal{X}\tilde{\otimes}\mathcal{Y})_{abcd} := \frac{1}{2}\mathcal{X}_{ab}\mathcal{Y}_{cd} + \frac{1}{2}\mathcal{Y}_{ab}\mathcal{X}_{cd} - \frac{1}{3}\mathcal{I}_{abcd}\mathcal{X}_{ef}\mathcal{Y}^{ef} \ , \tag{2.5}$$

a symmetric bilinear product taking two anti-self-dual forms to a complex $(0,4)$-tensor. It is simple to verify that such a tensor automatically satisfies the algebraic symmetries of the Weyl conformal tensor: i) it is antisymmetric in its first two, and last two, indices $(\mathcal{X}\tilde{\otimes}\mathcal{Y})_{abcd} = -(\mathcal{X}\tilde{\otimes}\mathcal{Y})_{bacd} = -(\mathcal{X}\tilde{\otimes}\mathcal{Y})_{abdc}$ ii) it is symmetric swapping the first two and the last two sets of indices $(\mathcal{X}\tilde{\otimes}\mathcal{Y})_{abcd} = (\mathcal{X}\tilde{\otimes}\mathcal{Y})_{cdab}$ iii) it verifies the first Bianchi identity $(\mathcal{X}\tilde{\otimes}\mathcal{Y})_{abcd} + (\mathcal{X}\tilde{\otimes}\mathcal{Y})_{bcad} + (\mathcal{X}\tilde{\otimes}\mathcal{Y})_{cabd} = 0$ and iv) it is trace-free $(\mathcal{X}\tilde{\otimes}\mathcal{Y})_{abcd}g^{ac} = 0$. For lack of a better name, this product will be referred to as a *symmetric spinor product*, using the fact that in a representation using spinor coördinates $\mathcal{X}_{ab} = f_{AB}\epsilon_{A'B'}$ and $\mathcal{Y}_{ab} = h_{AB}\epsilon_{A'B'}$ (where $f_{AB} = f_{BA}$, and similarly for h_{AB}), the product can be written as

$$(\mathcal{X}\tilde{\otimes}\mathcal{Y})_{abcd} \propto f_{(AB}h_{CD)}\epsilon_{A'B'}\epsilon_{C'D'} \ ,$$

where $(\cdot)$ denotes complete symmetrization of the indices. Notice that by definition

$$(\mathcal{P}_-(\mathcal{X}\tilde{\otimes}\mathcal{Y})\mathcal{P}_-)_{abcd} = (\mathcal{X}\tilde{\otimes}\mathcal{Y})_{abcd} \ .$$

2.1.2 Complex curvature tensor

In view of Theorem 2.1.2, the derivations in Section 2.1.1 naturally leads to the notion of a complex curvature tensor. Consider $\mathcal{X} \in \Lambda_-$, the identity

$$-iR^{(j)}_{abcd}\mathcal{X}^{cd} = R^{(j)}_{abcd}{}^*\mathcal{X}^{cd} = \frac{(-1)^{\delta_{3j}}}{2}\epsilon_{abcd}R^{(j)cdef}\mathcal{X}_{ef} = (-1)^{\delta_{3j}}{}^*(R^{(j)}\mathcal{X})_{ab}$$

gives that $R^{(j)}_{abcd}$ maps $\Lambda_+ \to \Lambda_+$ and $\Lambda_- \to \Lambda_-$ if $j \in \{1,2,4\}$; and $\Lambda_- \to \Lambda_+$ and vice versa if $j = 3$.

Hence the following decomposition of the Riemann curvature tensor relative to the eigenspaces of $*$ is obtained:

$$R_{abcd} = (\mathcal{P}_-R\mathcal{P}_-)_{abcd} + (\mathcal{P}_+R\mathcal{P}_+)_{abcd} + (\mathcal{P}_-R\mathcal{P}_+)_{abcd} + (\mathcal{P}_+R\mathcal{P}_-)_{abcd} \ ,$$

where the first two terms form $R^{(1)}_{abcd} + R^{(2)}_{abcd} + R^{(4)}_{abcd}$ and the last two terms form $R^{(3)}_{abcd}$. From the coördinate expressions of $\mathcal{P}_\pm$ in terms of $\mathcal{I}_{abcd}$, it is clear that the first two terms are complex conjugates of each other, and similarly the last two terms. A simple computation shows that, in terms of (2.1),

$$(\mathcal{P}_- R \mathcal{P}_-)_{abcd} = \frac{1}{2}(W_{abcd} + \frac{i}{2}\epsilon_{abef}W^{ef}{}_{cd}) + \frac{R}{12}\mathcal{I}_{abcd} \ ,$$
$$(\mathcal{P}_+ R \mathcal{P}_-)_{abcd} = \frac{1}{4}[(g \owedge S)_{abcd} + i(S_a{}^e\epsilon_{ebcd} + S_b{}^f\epsilon_{afcd})] \ .$$

In the sequel, $\mathcal{C}_{abcd}$ will be used to denote the complex Weyl tensor

$$\mathcal{C}_{abcd} := \frac{1}{2}(W_{abcd} + \frac{i}{2}\epsilon_{abef}W^{ef}{}_{cd}) \ . \tag{2.6}$$

Now, in the case of the Einstein-Maxwell equations, a solution must satisfy $R = 0$ and $S_{ab} = R_{ab} = T_{ab}$. The tensor

$$\mathcal{E}_{abcd} := \frac{1}{4}[(g \owedge T)_{abcd} + i(T_a{}^e\epsilon_{ebcd} + T_b{}^f\epsilon_{afcd})] \tag{2.7}$$

together with $\mathcal{C}_{abcd}$ completely specifies the Riemann curvature tensor. Note also that in terms of the complexified Maxwell field $\mathcal{H}_{ab} = \frac{1}{2}(H_{ab} + i^*H_{ab})$ the stress-energy tensor can be written as

$$T_{ab} = 4\mathcal{H}_{ac}\bar{\mathcal{H}}_b{}^c = 4\mathcal{H}_b{}^c\bar{\mathcal{H}}_{ac} \ . \tag{2.8}$$

It is with this form most of the subsequent computations will be made.

2.2 Killing symmetry

Given $(\mathcal{M}, g_{ab})$ a smooth, four-dimensional Lorentzian manifold, and assuming that it admits a smooth Killing vector field t^a, the Ernst two-form can be defined by

$$F_{ab} = \nabla_a t_b - \nabla_b t_a = 2\nabla_a t_b \ , \tag{2.9}$$

the second equality being a consequence of the Killing equation. As is well-known, the Ernst two-form satisfies

$$\nabla_c F_{ab} = 2\nabla_c \nabla_a t_b = 2R_{dcab} t^d \ . \tag{2.10}$$

This directly implies a divergence-curl system (in other words, a Maxwell equation with source terms) satisfied by the two-form

$$\nabla_{[c} F_{ab]} = 0 \ ,$$
$$\nabla^a F_{ab} = -2R_{db} t^d \ .$$

Here one of the primary differences of the present work from [22] is seen: a space-time satisfying the Einstein vacuum equations is Ricci-flat, and the above implies that the Ernst two-form satisfies the sourceless Maxwell equations. In particular, for the vacuum case,

$$\nabla_{[c} \mathcal{F}_{ab]} = 0 \ ,$$

and a calculation then verifies that

$$\nabla_{[c} (\mathcal{F}_{a]b} t^b) = 0 \ .$$

Thus an Ernst potential σ is constructed for

$$\nabla_a \sigma = \mathcal{F}_{ab} t^b$$

if the space-time is assumed to be simply connected.

In the non-vacuum case that this paper deals with, this construction cannot be exactly carried through. However, the essence of the construction above is the following fact disjoint from the semi-Riemannian structure of our setup: consider a smooth manifold $\mathcal{M}$, a smooth differential form X, and a smooth vector-field v. Consider the Cartan relation

$$\mathcal{L}_v X = i_v \circ dX + d \circ i_v X$$

where $\mathcal{L}_v$ stands for the Lie derivative relative to the vector-field v, and i_v is the interior derivative. If X is a closed form, and v is a symmetry of X (i.e. $\mathcal{L}_v X = 0$), then $i_v X$ must be closed also.

Applying to the Einstein-Maxwell equations, take X to be the anti-self-dual Maxwell form

$$\mathcal{H}_{ab} := \frac{1}{2}(H_{ab} + i^* H_{ab}) \ , \tag{2.11}$$

which by Maxwell's equations is closed. The vector-field v is naturally the Killing field t^a, and therefore the complex-valued one-form $\mathcal{H}_{ab} t^a$ is closed, and if $\mathcal{M}$ is taken to be simply connected, also exact. In the sequel the complex-valued function Ξ, which is defined by

$$\nabla_b \Xi = \mathcal{H}_{ab} t^a \ , \tag{2.12}$$

will be used. Notice that *a priori* Ξ is only defined up to the addition of a constant. If the space-time is assumed to be also asymptotically flat, Ξ can be uniquely normalized by a decay condition $\Xi \to 0$ at spatial infinity (see Section 3.1 for more detail). The function Ξ takes the place of the Ernst potential σ used in [22].

2.3 Some computations regarding the main players

The Ernst two-form, the Maxwell two-form, and the Weyl curvature are the principal players in many of the computations in subsequent chapters. Here some of their properties, all related to the divergence-curl system they satisfy, are recorded.

Consider the decomposition of the Riemann curvature tensor

$$(R\mathcal{P}_-)_{abcd} = \mathcal{C}_{abcd} + \mathcal{E}_{abcd} \ .$$

Because we act by $\mathcal{P}_-$ on the right (and not on the left), the expression still observes the second Bianchi identity

$$\nabla_{[e}(\mathcal{C} + \mathcal{E})_{ab]cd} = 0 \ . \tag{2.13}$$

Take a contraction between the indices e, c,

$$\nabla^c \mathcal{C}_{abcd} + \nabla^c \mathcal{E}_{abcd} = \frac{1}{2}(\nabla_a T_{bd} - \nabla_b T_{ad}) \ .$$

Noting that

$$\mathcal{E}_{abcd} = \frac{1}{2}(g \owedge T)_{ab}{}^{ef}\mathcal{I}_{efcd}) = \frac{1}{2}\bar{\mathcal{I}}_{abef}(g \owedge T)^{ef}{}_{cd} = \bar{\mathcal{E}}_{cdab}$$

by the second Singer-Thorpe theorem,

$$\nabla^c \mathcal{E}_{abcd} = \bar{\mathcal{I}}_{abef}(\nabla^e T^f_d) \ . \tag{2.14}$$

Using that $(\mathcal{I} + \bar{\mathcal{I}})_{abef} X^{ef} = X_{ab}$, we obtain the contracted second Bianchi identity

$$\nabla^c \mathcal{C}_{abcd} = \mathcal{I}_{abef} \nabla^e T^f_d \ . \tag{2.15}$$

From the Maxwell equations, $\mathcal{H}_{ab}$ is harmonic. Combining the two gives

$$\begin{aligned} 0 &= \nabla^c[\nabla_c\mathcal{H}_{ab} + \nabla_a\mathcal{H}_{bc} + \nabla_b\mathcal{H}_{ca}] \\ \Box_g\mathcal{H}_{ab} &= [\nabla^c, \nabla_a]\mathcal{H}_{cb} - [\nabla^c, \nabla_b]\mathcal{H}_{ca} \qquad (2.16) \\ &= R^c{}_{ac}{}^d\mathcal{H}_{db} + R^c{}_{ab}{}^d\mathcal{H}_{cd} - R^c{}_{bc}{}^d\mathcal{H}_{da} - R^c{}_{ba}{}^d\mathcal{H}_{cd} \\ &= T_{bd}g_{ac}\mathcal{H}^{cd} - T_{ad}g_{bc}\mathcal{H}^{cd} - R_{abcd}\mathcal{H}^{cd} \\ &= -W_{abcd}\mathcal{H}^{cd} = -\mathcal{C}_{abcd}\mathcal{H}^{cd} \ . \qquad (2.17) \end{aligned}$$

The derivative of $\mathcal{F}_{ab}$ can be written down explicitly as

$$\begin{aligned} \nabla_c\mathcal{F}_{ab} &= (R_{dcab} + iR^*_{dcab})t^d \\ &= 2\mathcal{C}_{dcab}t^d + 2\mathcal{E}_{dcab}t^d \ . \qquad (2.18) \end{aligned}$$

Taking the trace gives immediately

$$\nabla^a\mathcal{F}_{ab} = -T_{ab}t^a \ .$$

On the other hand, notice that

$$\nabla^a(\bar{\Xi}\mathcal{H}_{ab}) = -\bar{\mathcal{H}}^{ad}t_d\mathcal{H}_{ab} = -\frac{1}{4}T_{bd}t^d \ ,$$

which implies

$$\nabla^a(\mathcal{F}_{ab} - 4\bar{\Xi}\mathcal{H}_{ab}) = 0 \ . \qquad (2.19)$$

Since $\mathcal{F}_{ab} - 4\bar{\Xi}\mathcal{H}_{ab}$ is anti-self-dual, the fact it is divergence free implies that it is also curl free, and hence it is a Maxwell field.

The following fact about Killing vector fields will also be needed. Consider the product ${}^*F_{ab}{}^*F_{cd} = \frac{1}{4}\epsilon_{abef}\epsilon_{cdgh}F^{ef}F^{gh}$. The product of the Levi-Civita symbols can be expanded in terms of the metric:

$$\epsilon_{ijkl}\epsilon^{qrst} = -24 g_i^{[q} g_j^r g_k^s g_l^{t]} \ .$$

By explicit computation using this expansion,

$$\begin{aligned} {}^*F_{mx}t^{x*}F_{ny}t^y = \frac{1}{2}F_{ab}F^{ab}(t_m t_n - t_x t^x g_{mn}) + g_{mn}F_{xa}t^x F^{ya}t_y - F_{nx}t^x F_{my}t^y \\ + F^{bx}t_x t_m F_{nb} + F^{bx}t_x t_n F_{mb} + t_x t^x F_{ma}F_n{}^a \ . \end{aligned}$$

Writing $t^2 = t_a t^a$, from the fact $\nabla_b t^2 = t^a F_{ba}$ the following, which is identical to equation (13) from [22], is obtained:

$$\begin{aligned} {}^*F_{mx}t^{x*}F_{ny}t^y = \frac{1}{2}F_{ab}F^{ab}(t_m t_n - g_{mn}t^2) + g_{mn}\nabla_a t^2 \nabla^a t^2 - \nabla_m t^2 \nabla_n t^2 \\ + t_m F_{nb}\nabla^b t^2 + t_n F_{mb}\nabla^b t^2 + t^2 F_{ma}F_n{}^a \ . \end{aligned} \quad (2.20)$$

2.4 Tetrad formalism

The null tetrad formalism of Newman and Penrose will be used extensively in the calculations below, albeit with slightly different notational conventions. In the following, a dictionary is given between the standard Newman-Penrose variables (see, e.g. Chapter 7 in [39]) and the null-structure variables of Ionescu and Klainerman [14] which is used in this work.

Following Ionescu and Klainerman [14], the space-time is assumed to contain a natural choice of a *null pair* $\{\underline{l}, l\}$. Recall that the complex valued vector field m is said to be *compatible* with the null pair if

$$g(l,m) = g(\underline{l},m) = g(m,m) = 0 \ , \quad g(m,\bar{m}) = 1$$

where $\bar{m}$ is the complex conjugate of m. Given a null pair, for any point $p \in \mathcal{M}$, such a compatible vector field always exist on a sufficiently small neighborhood of p. The set of vector fields $\{m, \bar{m}, \underline{l}, l\}$ is said to form a *null tetrad* if, in addition, they have positive orientation $\epsilon_{abcd} m^a \bar{m}^b \underline{l}^c l^d = i$ (m and $\bar{m}$ can always be swapped by the obvious transformation to satisfy this condition).

The scalar functions corresponding to the connection coefficients of of the null tetrad are defined, with translation to the Newman-Penrose formalism, in Table 2.1. The Γ-notation is defined by

$$\Gamma_{\alpha\beta\gamma} = g(\nabla_{e_\gamma} e_\beta, e_\alpha)$$

where for $e_1 = m$, $e_2 = \bar{m}$, $e_3 = \underline{l}$, and $e_4 = l$. It is clear that $\Gamma_{(\alpha\beta)\gamma} = 0$, i.e. it is antisymmetric in the first two indices. Two natural[2] operations are then defined: the under-bar (e.g. $\theta \leftrightarrow \underline{\theta}$) corresponds to swapping the indices $3 \leftrightarrow 4$ (e.g. $\Gamma_{142} \leftrightarrow \Gamma_{132}$), and complex conjugation (e.g. $\theta \leftrightarrow \bar{\theta}$) corresponds to swapping the numeric indices $1 \leftrightarrow 2$ (e.g. $\Gamma_{142} \leftrightarrow \Gamma_{241}$). Note that $\theta, \underline{\theta}, \vartheta, \underline{\vartheta}, \xi, \underline{\xi}, \eta, \underline{\eta}, \zeta$ are complex-valued, while ω and $\underline{\omega}$ are real-valued; thus the connection-coefficients defined in Table 2.1, along with their complex conjugates, define 20 out of the 24 rotation coefficients: the only ones not given a "name" are $\Gamma_{121}, \Gamma_{122}, \Gamma_{123}, \Gamma_{124}$, among which the first two are related by complex-conjugation, and the latter-two by under-bar.

[2]Buyers beware: the operations are only natural in so much as those geometric statements that are agnostic to orientation of the frame vectors. Indeed, both the under-bar and complex conjugation changes the sign of the Levi-Civita symbol; while for the complex conjugation it is of less consequence (since the complex conjugate of $-i$ is i, the sign difference is most naturally absorbed), for the under-bar operation one needs to take care in application to ascertain that sign-changes due to, say, the Hodge star operator is not present in the equation under consideration. In particular, generally coördinate independent geometric statements (such as the relations to be developed in this section) will be compatible with consistent application of the under-bar operations, while statements dependent on a particular choice of foliation or frame will usually need to be evaluated on a case-by-case basis.

	Γ-notation	Newman-Penrose	Ionescu-Klainerman
$g(\nabla_{\bar{m}} l, m)$	Γ_{142}	$-\rho$	θ
$g(\nabla_{\bar{m}} \underline{l}, m)$	Γ_{132}	$\bar{\mu}$	$\underline{\theta}$
$g(\nabla_m l, m)$	Γ_{141}	$-\sigma$	ϑ
$g(\nabla_m \underline{l}, m)$	Γ_{131}	$\bar{\lambda}$	$\underline{\vartheta}$
$g(\nabla_l l, m)$	Γ_{144}	$-\kappa$	ξ
$g(\nabla_{\underline{l}} \underline{l}, m)$	Γ_{133}	$\bar{\nu}$	$\underline{\xi}$
$g(\nabla_{\underline{l}} l, m)$	Γ_{143}	$-\tau$	η
$g(\nabla_l \underline{l}, m)$	Γ_{134}	$\bar{\pi}$	$\underline{\eta}$
$g(\nabla_l l, \underline{l})$	Γ_{344}	$-2\epsilon + \Gamma_{214}$	ω
$g(\nabla_{\underline{l}} \underline{l}, l)$	Γ_{433}	$2\gamma + \Gamma_{123}$	$\underline{\omega}$
$g(\nabla_m l, \underline{l})$	Γ_{341}	$-2\beta + \Gamma_{211}$	$\zeta = -\underline{\zeta}$

Table 2.1: Dictionary of Ricci rotation coefficients vs. Newman-Penrose spin coefficients vs. Ionescu-Klainerman connection coefficients

The directional derivative operators are given by:

$$D = l^a \nabla_a \ , \ \underline{D} = \underline{l}^a \nabla_a \ , \ \delta = m^a \nabla_a \ , \ \bar{\delta} = \bar{m}^a \nabla_a$$

(their respective symbols in Newman-Penrose notation are $D, \Delta, \delta, \bar{\delta}$).

The spinor components of the Riemann curvature tensor can be given in terms of the following: let W_{abcd} be the Weyl curvature tensor, S_{ab} be the traceless Ricci tensor, and R be the scalar curvature,

$$\Psi_2 = W(l, m, l, m) \tag{2.21a}$$

$$\bar{\Psi}_{-2} = \underline{\Psi}_2 = W(\underline{l}, m, \underline{l}, m) \tag{2.21b}$$

$$\Psi_1 = W(m, l, \underline{l}, l) \tag{2.21c}$$

$$\bar{\Psi}_{-1} = \underline{\Psi}_1 = W(m, \underline{l}, l, \underline{l}) \tag{2.21d}$$

$$\Psi_0 = W(\bar{m}, \underline{l}, m, l) \tag{2.21e}$$

$$\Phi_{11} = S(l, l) \tag{2.21f}$$

$$\underline{\Phi}_{11} = S(\underline{l}, \underline{l}) \tag{2.21g}$$

$$\Phi_{01} = S(m, l) \tag{2.21h}$$

$$\underline{\Phi}_{01} = S(m, \underline{l}) \tag{2.21i}$$

$$\Phi_{00} = S(m, m) \tag{2.21j}$$

$$\Phi_{0} = \frac{1}{2}[S(l, \underline{l}) + S(m, \bar{m})] \tag{2.21k}$$

Notice that the quantities Ψ_A, $A \in \{-2, -1, 0, 1, 2\}$ are automatically anti-self-dual: replacing $W_{abcd} \leftrightarrow {}^*W_{abcd}$ gives $\Psi_A({}^*W) = (-i)\Psi_A(W)$, which follows from the orthogonality properties of the null tetrad and the orientation $\epsilon(m, \bar{m}, \underline{l}, l) = i$. Using this notation, the *null structure equations*, which are equivalent to the Newman-Penrose equations, can be derived from the definition of the Riemann curvature tensor:

$$R_{\alpha\beta\mu\nu} = e_\mu(\Gamma_{\alpha\beta\nu}) - e_\nu(\Gamma_{\alpha\beta\mu}) + \Gamma^\rho{}_{\beta\nu}\Gamma_{\alpha\rho\mu} - \Gamma^\rho{}_{\beta\mu}\Gamma_{\alpha\rho\nu} + (\Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu})\Gamma_{\alpha\beta\rho}$$

and that

$$R_{\alpha\beta\mu\nu} = W_{\alpha\beta\mu\nu} + \frac{1}{2}(S_{\alpha\mu}g_{\beta\nu} + S_{\beta\nu}g_{\alpha\mu} - S_{\alpha\nu}g_{\beta\mu} - S_{\beta\mu}g_{\alpha\nu}) + \frac{1}{12}R(g_{\alpha\mu}g_{\beta\nu} - g_{\beta\mu}g_{\alpha\nu}) \ .$$

So from $R_{1441} = W_{1441} = -\Psi_2$,

$$(D + 2\Gamma_{124})\vartheta - (\delta + \Gamma_{121})\xi = \xi(2\zeta + \eta + \underline{\eta}) - \vartheta(\omega + \theta + \bar{\theta}) - \Psi_2 \ , \tag{2.22a}$$

and by taking under-bar of the whole expression, a similar expression for $R_{1331} = -\underline{\Psi}_2$ can be had (in the interest of space, the obvious changes of variables are omitted here).

For $R_{1442} = -\frac{1}{2}S_{44}$ (and analogously $R_{1332} = -\frac{1}{2}S_{33}$),

$$D\theta - (\bar{\delta} + \Gamma_{122})\xi = -\theta^2 - \omega\theta - \vartheta\bar{\vartheta} + \bar{\xi}\eta + \xi(2\bar{\zeta} + \underline{\bar{\eta}}) - \frac{1}{2}\Phi_{11} \ . \tag{2.22b}$$

From $R_{1443} = -\Psi_1 - \frac{1}{2}S_{14}$,

$$(D + \Gamma_{124})\eta - (\underline{D} + \Gamma_{123})\xi = -2\underline{\omega}\xi + \theta(\underline{\eta} - \eta) + \vartheta(\underline{\bar{\eta}} - \bar{\eta}) - \Psi_1 - \frac{1}{2}\Phi_{01} \ . \tag{2.22c}$$

From $R_{1431} = \frac{1}{2}S_{11}$,

$$(\underline{D} + 2\Gamma_{123})\vartheta - (\delta + \Gamma_{121})\eta = \eta^2 + \xi\underline{\xi} - \theta\underline{\vartheta} + \vartheta(\underline{\omega} - \underline{\bar{\theta}}) + \frac{1}{2}\Phi_{00} \ . \tag{2.22d}$$

From $R_{1432} = -\Psi_0 + \frac{1}{12}R$,

$$\underline{D}\theta - (\bar{\delta} + \Gamma_{122})\eta = \xi\underline{\bar{\xi}} + \eta\bar{\eta} - \vartheta\underline{\bar{\vartheta}} + \theta(\underline{\omega} - \underline{\theta}) - \Psi_0 + \frac{R}{12} \ . \tag{2.22e}$$

From $R_{1421} = -\Psi_1 + \frac{1}{2}S_{41}$,

$$(\bar{\delta} + 2\Gamma_{122})\vartheta - \delta\theta = \zeta\theta - \bar{\zeta}\vartheta + \eta(\theta - \bar{\theta}) + \xi(\underline{\theta} - \underline{\bar{\theta}}) - \Psi_1 + \frac{1}{2}\Phi_{01} \ . \tag{2.22f}$$

Using $R_{3441} = -\Psi_1 - \frac{1}{2}S_{41}$,

$$(D+\Gamma_{124})\zeta - \delta\omega = \omega(\zeta + \underline{\eta}) + \bar{\theta}(\underline{\eta} - \zeta) + \vartheta(\underline{\bar{\eta}} - \bar{\zeta}) - \xi(\underline{\bar{\theta}} + \underline{\omega}) - \bar{\xi}\underline{\vartheta} - \Psi_1 - \frac{1}{2}\Phi_{01} \ . \tag{2.22g}$$

From $R_{3443} = \Psi_0 + \bar{\Psi}_0 - S_{34} + \frac{R}{12}$,

$$D\underline{\omega} + \underline{D}\omega = \bar{\xi}\underline{\xi} + \xi\underline{\bar{\xi}} - \bar{\eta}\underline{\eta} - \eta\underline{\bar{\eta}} + \zeta(\bar{\eta} - \underline{\bar{\eta}}) + \bar{\zeta}(\eta - \underline{\eta}) - (\Psi_0 + \bar{\Psi}_0) + \Phi_0 - \frac{R}{12} \ . \tag{2.22h}$$

And lastly from $R_{3421} = \Psi_0 - \bar{\Psi}_0$,

$$(\delta - \Gamma_{121})\bar{\zeta} - (\bar{\delta} + \Gamma_{122})\zeta = (\bar{\vartheta}\underline{\vartheta} - \vartheta\underline{\bar{\vartheta}}) + (\theta\underline{\bar{\theta}} - \bar{\theta}\underline{\theta}) + \underline{\omega}(\theta - \bar{\theta}) - \omega(\underline{\theta} - \underline{\bar{\theta}}) - (\Psi_0 - \bar{\Psi}_0)\ . \quad (2.22i)$$

The Maxwell equations can also be decomposed in this formalism: let

$$\Upsilon_0 = \frac{1}{2}(H(l, \underline{l}) + H(\bar{m}, m)) = \mathcal{H}_{ab} l^a \underline{l}^b \quad (2.23a)$$

$$\Upsilon_1 = H(l, m) = \mathcal{H}_{ab} l^a m^b \quad (2.23b)$$

$$\bar{\Upsilon}_{-1} = \underline{\Upsilon}_1 = H(m, \underline{l}) = \bar{\mathcal{H}}_{ab} m^a \underline{l}^b \quad (2.23c)$$

be the spinor components of the Maxwell two-form H_{ab}. Maxwell's equations become

$$\underline{D}\Upsilon_0 - (\delta - \Gamma_{121})\Upsilon_{-1} = \underline{\bar{\xi}}\Upsilon_1 - 2\underline{\bar{\theta}}\Upsilon_0 - (\zeta - \eta)\Upsilon_{-1} \quad (2.24a)$$

$$(\underline{D} + \Gamma_{123})\Upsilon_1 - \delta\Upsilon_0 = (\underline{\omega} - \underline{\bar{\theta}})\Upsilon_1 + 2\eta\Upsilon_0 - \vartheta\Upsilon_{-1} \quad (2.24b)$$

and their under-bar counterparts.

The Bianchi identities

$$\nabla_{[e} R_{ab]cd} = 0$$

also need to be expressed in the formalism. Note that this implies

$$\nabla^e W_{ebcd} = \nabla_{[c} S_{d]b} - \frac{1}{12} g_{b[c} \nabla_{d]} R =: J_{bcd}\ ,$$

which gives

$$\nabla_{[e} W_{ab]cd} = \frac{1}{6} \epsilon_{seab} J^{srt} \epsilon_{rtcd}\ .$$

Using the orientation condition $\epsilon(m, \bar{m}, \underline{l}, l) = i$, the following can be demonstrated

$$(\bar{\delta} + 2\Gamma_{122})\Psi_2 - (D + \Gamma_{124})\Psi_1 + \frac{1}{2}\delta\Phi_{11} - \frac{1}{2}(D + \Gamma_{124})\Phi_{01} \tag{2.25a}$$
$$= -(2\bar{\zeta} + \bar{\underline{\eta}})\Psi_2 + (4\theta + \omega)\Psi_1 + 3\xi\Psi_0$$
$$- (\bar{\theta} + \frac{1}{2}\omega)\Phi_{01} - \vartheta\bar{\Phi}_{01} + (\zeta + \frac{1}{2}\underline{\eta})\Phi_{11} + \xi\Phi_0 + \frac{1}{2}\bar{\xi}\Phi_{00}$$

$$(\underline{D} + 2\Gamma_{123})\Psi_2 - (\delta + \Gamma_{121})\Psi_1 + \frac{1}{2}(D + 2\Gamma_{124})\Phi_{00} - \frac{1}{2}(\delta + \Gamma_{121})\Phi_{01} \tag{2.25b}$$
$$= (2\underline{\omega} - \bar{\underline{\theta}})\Psi_2 + (\zeta + 4\eta)\Psi_1 + 3\vartheta\Psi_0$$
$$- \frac{1}{2}\bar{\theta}\Phi_{00} - \vartheta\Phi_0 - \frac{1}{2}\underline{\vartheta}\Phi_{11} + \xi\underline{\Phi}_{01} + (\frac{1}{2}\zeta + \underline{\eta})\Phi_{01}$$

$$-(\bar{\delta} + \Gamma_{122})\Psi_1 - D\Psi_0 - \frac{1}{2}D\Phi_0 + \frac{1}{2}(\delta - \Gamma_{121})\bar{\Phi}_{01} - \frac{1}{24}DR \tag{2.25c}$$
$$= -\bar{\underline{\vartheta}}\Psi_2 + (2\bar{\underline{\eta}} + \bar{\zeta})\Psi_1 + 3\theta\Psi_0 + 2\xi\bar{\underline{\Psi}}_1$$
$$- \frac{1}{2}(\zeta + \underline{\eta})\bar{\Phi}_{01} + \bar{\theta}\Phi_0 + \frac{1}{2}\bar{\underline{\theta}}\Phi_{11} + \frac{1}{2}\vartheta\bar{\Phi}_{00}$$
$$- \frac{1}{2}\bar{\xi}\underline{\Phi}_{01} - \frac{1}{2}\bar{\underline{\eta}}\Phi_{01} - \frac{1}{2}\xi\bar{\underline{\Phi}}_{01}$$

$$(D + \Gamma_{124})\underline{\Psi}_1 + \delta\bar{\Psi}_0 + \frac{1}{2}(\underline{D} + \Gamma_{123})\Phi_{01} - \frac{1}{2}\delta\Phi_0 + \frac{1}{24}\delta R \tag{2.25d}$$
$$= -2\underline{\vartheta}\bar{\Psi}_1 - 3\underline{\eta}\bar{\Psi}_0 + (\omega - 2\bar{\theta})\underline{\Psi}_1 + \bar{\xi}\underline{\Psi}_2$$
$$+ \frac{1}{2}(\underline{\omega} - \bar{\underline{\theta}})\Phi_{01} - \frac{1}{2}\bar{\theta}\underline{\Phi}_{01} - \frac{1}{2}\vartheta\bar{\underline{\Phi}}_{01} - \frac{1}{2}\underline{\vartheta}\bar{\Phi}_{01}$$
$$+ \frac{1}{2}\bar{\eta}\Phi_{00} + \eta\Phi_0$$

In addition, taking the trace of the Bianchi identities gives

$$0 = \nabla^e W_{ebc}{}^b = J_{bc}{}^b$$

and evaluates to

$$
\begin{aligned}
&-\delta\Phi_0 - (\bar{\delta} + 2\Gamma_{122})\Phi_{00} + (\underline{D} + \Gamma_{123})\Phi_{01} + (D + \Gamma_{124})\underline{\Phi}_{01} + \frac{1}{4}\delta R \qquad (2.25e)\\
&\quad = (\bar{\eta} + \underline{\bar{\eta}})\Phi_{00} + 2(\eta + \underline{\eta})\Phi_0 + (\omega - 2\theta - \bar{\theta})\underline{\Phi}_{01} + (\underline{\omega} - 2\underline{\theta} - \underline{\bar{\theta}})\Phi_{01}\\
&\qquad - \vartheta\underline{\bar{\Phi}}_{01} - \underline{\vartheta}\bar{\Phi}_{01} + \xi\underline{\Phi}_{11} + \underline{\xi}\Phi_{11}\\
&D\Phi_0 + \underline{D}\Phi_{11} - (\delta - \Gamma_{121})\bar{\Phi}_{01} - (\bar{\delta} + \Gamma_{122})\Phi_{01} + \frac{1}{4}DR \qquad (2.25f)\\
&\quad = -\bar{\vartheta}\Phi_{00} - 2(\bar{\theta} + \theta)\Phi_0 + \bar{\xi}\underline{\Phi}_{01} + (\bar{\zeta} + 2\bar{\eta} + \underline{\bar{\eta}})\Phi_{01} - \vartheta\bar{\Phi}_{00}\\
&\qquad + \xi\underline{\bar{\Phi}}_{01} + (\zeta + 2\eta + \underline{\eta})\bar{\Phi}_{01} + (2\underline{\omega} - \underline{\theta} - \underline{\bar{\theta}})\Phi_{11}
\end{aligned}
$$

A simple identification using Table 2.1 and the definitions for various spinor components of the Riemann and traceless Ricci tensors shows that one can recover all of the Bianchi identities in Newman-Penrose formalism from the above six equations through the action of complex-conjugation and under-barring.

Lastly, to complete the formalism, the commutator relations are recorded here:

$$[D, \underline{D}] = (\underline{\eta} - \eta)\bar{\delta} + (\underline{\bar{\eta}} - \bar{\eta})\delta - \underline{\omega}D + \omega\underline{D} \qquad (2.26a)$$

$$[D, \delta] = -\vartheta\bar{\delta} - (\Gamma_{124} + \bar{\theta})\delta + (\underline{\eta} + \zeta)D + \xi\underline{D} \qquad (2.26b)$$

$$[\delta, \bar{\delta}] = \Gamma_{121}\bar{\delta} + \Gamma_{122}\delta + (\underline{\bar{\theta}} - \underline{\theta})D + (\bar{\theta} - \theta)\underline{D} \qquad (2.26c)$$

2.5 Geometry of bifurcate event horizon

As discussed in Section 1.3, the properties of the bifurcate event horizon is well studied. Here we summarize some of the trivial geometric constructions related to it. Throughout $\mathfrak{H}^{\pm}$ will denote a pair of smooth null hypersurfaces of $(\mathcal{M}, g_{ab})$, intersecting transversely in $\mathfrak{H}_0$, a topological sphere. We will also require that $\mathfrak{H}^{\pm}$ have vanishing null mean curvature as required by Hawking's area theorem.

First recall the definition of the null mean curvature. Given $\mathfrak{H} \subset \mathcal{M}$ a smooth null hypersurface in a Lorentzian manifold, it is easily verified that at any point $p \in \mathfrak{H}$ there exists a unique direction $L \in T_p\mathfrak{H} \subset T_p\mathcal{M}$ such that $g(L, v) = 0$ for any $v \in T_p\mathfrak{H}$. By suitably normalizing L we can require it to be a smooth, future-pointing, null geodesic vector field tangent to $\mathfrak{H}$; the geodesics tangent to L are said to be the generators of the hypersurface $\mathfrak{H}$. We can define a horizontal structure ${}^hT\mathfrak{H} = T\mathfrak{H}/L$ by identifying two elements of $T_p\mathfrak{H}$ when they differ by a factor of L. The metric on $\mathcal{M}$ induces a Riemannian metric on ${}^hT\mathfrak{H}$. The null Weingarten map associated to L is defined to be $b_L : {}^hT\mathfrak{H} \to {}^hT\mathfrak{H}$ given by

$$b_L([X]) = [\nabla_X L]$$

where X is a representative in $T\mathfrak{H}$ for $[X] \in {}^hT\mathfrak{H}$. It is easily checked that since $g(\nabla_X L, L) = \frac{1}{2}\partial_X g(L, L) = 0$, $\nabla_X L \in T\mathfrak{H}$. A simple computation shows the null Weingarten map is well defined for a fixed L, and that it is tensorial in L (namely $b_{fL} = f b_L$). The null mean curvature for $\mathfrak{H}$ relative to L is defined as the trace of b_L. Notice that while the null mean curvature depends on the choice of the vector field L, its sign is invariant relative to the normalization of L. The null mean curvature is related to the area in the following way: let $S \subset \mathfrak{H}$ be a space-like hypersurface, and write ω_S for the induced volume form by the global metric g. Writing the one parameter family of flows generated by L as Φ_t, we have that

$$\partial_t(\Phi_{t*}\omega_S) = (tr\, b_L)\omega_S \ .$$

In other words, the null mean curvature measures the growth of the volume form between successive spatial slices of $\mathfrak{H}$, which is why in the physics literature it is also known as the null expansion for a null congruence. In particular, a null hypersurface for which the null mean curvature vanishes is said to be *non-expanding*.

2.5.1 Non-expanding null hypersurfaces and adapted tetrads

Now, consider a tetrad $m, \bar{m}, \underline{l}, l$ adapted to the null hypersurface $\mathfrak{H}$ by requiring l to be the null generator of $\mathfrak{H}$ and that $\nabla_l l = 0$. Then the requirement that l is geodesic on $\mathfrak{H}$ translates to the requirement that the null structure coefficients $\xi = \omega = 0$ on $\mathfrak{H}$. If we further require the condition that $m, \bar{m}$ are "tangent" to $\mathfrak{H}$, we see that the vanishing of null mean curvature is identical to requiring $\theta = 0$. (By fixing m and l, we also fix $\underline{l}$ by requiring it to be future pointing, orthogonal to m, and to have fixed inner product against l.)

Let us now consider the consequences of vanishing θ, ω, ξ by looking at the null structure equations.

- By (2.22b), which is in fact the Raychaudhuri equation in disguise, we have $|\vartheta|^2 + \frac{1}{2}\Phi_{11} = 0$. Suppose our space-time verifies the strong energy condition (which is satisfied by the Einstein-Maxwell system), $\Phi_{11} \geq 0$. So we can conclude that $\vartheta = 0$ and $\Phi_{11} = 0$. Now, since we are interested in the case of the Einstein-Maxwell system, we have

$$\Phi_{11} = S(l,l) = 4\mathcal{H}(l,m)\bar{\mathcal{H}}(l,\bar{m}) = 4|\Upsilon_1|^2$$

  and hence $\Upsilon_1 = 0$.

- By (2.22a), $\Psi_2 = 0$.

- By (2.22f), $\Psi_1 = \frac{1}{2}\Phi_{01}$. Notice that

$$\Phi_{01} = S(l,m) = 4\mathcal{H}(l,m)\bar{\mathcal{H}}(m,\bar{m}) = -4\Upsilon_1\Upsilon_0$$

  so $\Phi_{01} = 0$ from calculations above. And hence $\Psi_1 = 0$.

- By (2.22g), $(D + \Gamma_{124})\zeta = 0$. This says that ζ is essentially constant along generators of the null hypersurface.

- By (2.24a), $D\Upsilon_0 = 0$, so Υ_0 is constant along the generators of the null hypersurface.

- By (2.24b), $(D - \Gamma_{124})\Upsilon_{-1} = -(\bar{\delta} + 2\underline{\bar{\eta}})\Upsilon_0$.

- $\Phi_{00} = -4\mathcal{H}(l,m)\bar{\mathcal{H}}(m,\underline{l}) = -4\Upsilon_1\bar{\Upsilon}_{-1} = 0$.

- By (2.25c), $-D\Psi_0 - \frac{1}{2}D\Phi_0 = 0$ on $\mathfrak{H}$. Now $\Phi_0 = 4\Upsilon_0\bar{\Upsilon}_0$, so $D\Phi_0 = 0$, hence Ψ_0 is constant along generators of the null hypersurface.

Now, let $\mathfrak{H}^{\pm}$ be two smooth null hypersurfaces that intersect transversely in $\mathfrak{H}_0$. Let l be a geodesic generator of $\mathfrak{H}^{+}$, and $\underline{l}$ be a geodesic generator of $\mathfrak{H}^{-}$, normalized so that $g(l,\underline{l})|\mathfrak{H}_0 = -1$. We can complete a null tetrad along $\mathfrak{H}^{+} \cup \mathfrak{H}^{-}$ by requiring $m, \bar{m}$ tangent to $\mathfrak{H}_0$. If we assume that both $\mathfrak{H}^{+}$ and $\mathfrak{H}^{-}$ are non-expanding, the above analysis can be performed on both $\mathfrak{H}^{\pm}$ (with appropriate changes for $l \leftrightarrow \underline{l}$ and taking the underbar of all scalars defined in Section 2.4), and especially on $\mathfrak{H}_0$. We summarize the result here

- On $\mathfrak{H}^{+}$, $\theta = \omega = \xi = \vartheta = 0$, and $\Psi_2 = \Psi_1 = \Upsilon_1 = 0$.

- On $\mathfrak{H}^{-}$, $\underline{\theta} = \underline{\omega} = \underline{\xi} = \underline{\vartheta} = 0$, and $\Psi_{-2} = \Psi_{-1} = \Upsilon_{-1} = 0$.

- On $\mathfrak{H}^{+} \cup \mathfrak{H}^{-}$ Ψ_0 and Υ_0 are constant along the geodesic generators.

We also note that the Bianchi identities (2.25b, 2.25d) can be reduced to

$$(D - 2\Gamma_{124})\Psi_{-2} - (\bar{\delta} - \Gamma_{122})\Psi_{-1} - \frac{1}{2}(\bar{\delta} - \Gamma_{122})\underline{\bar{\Phi}}_{01} \tag{2.27a}$$

$$= (\underline{\bar{\zeta}} + 4\underline{\bar{\eta}})\Psi_{-1} + 3\underline{\bar{\vartheta}}\Psi_0 - \underline{\bar{\vartheta}}\Phi_0 + (\frac{1}{2}\underline{\bar{\zeta}} + \bar{\eta})\underline{\bar{\Phi}}_{01}$$

$$(D - \Gamma_{124})\Psi_{-1} + \bar{\delta}\Psi_0 + \frac{1}{2}(\underline{D} - \Gamma_{123})\bar{\Phi}_{01} - \frac{1}{2}\bar{\delta}\Phi_0 \tag{2.27b}$$

$$= -3\underline{\bar{\eta}}\Psi_0 + \bar{\eta}\Phi_0$$

2.5.2 Double null foliation near bifurcate sphere

The null tetrad chosen above still has considerable freedom in the gauge. In particular, in the construction above we can always locally modify m and $\bar{m}$ by factors of l (essentially picking different representatives in $T\mathfrak{H}$ of fixed elements in ${}^{h}T\mathfrak{H}$), while compensating in modifying the definition of $\underline{l}$. Here we'll present the double null foliation, which allows us to have a coördinate system in a neighborhood of $\mathfrak{H}_0$; the restriction of this coördinate system on $\mathfrak{H}^{\pm}$ gives a way to fix the null tetrad up to a complex rotation in m and $\bar{m}$. The construction is standard, and we follow here the presentation in [14].

The double null foliation is based on two optical functions $u, \underline{u}$ defined in a sufficiently small neighborhood of $\mathfrak{H}_0$ in $\mathcal{M}$. Let l be defined on $\mathfrak{H}^+$ and $\underline{l}$ be defined on $\mathfrak{H}^-$ as before by parallel translation. We define the functions $u, \underline{u}$ by setting $u = 0$ on $\mathfrak{H}^+$, and $\underline{u} = 0$ on $\mathfrak{H}^-$, and propagate u on $\mathfrak{H}^-$ by asking it to satisfy $\underline{l}(u) = -1$ (recall that $\underline{l}$ is future pointing, so u increases as we go into the past) and similarly propagate $\underline{u}$ on $\mathfrak{H}^+$ by $l(\underline{u}) = 1$. We get then a foliation of $\mathfrak{H}^-$ by the level sets $\mathfrak{H}_{u\underline{0}}$, and a foliation of $\mathfrak{H}^+$ by the level sets $\mathfrak{H}_{0\underline{u}}$. Then we can define l on $\mathfrak{H}^-$ as the unique future pointing null vector field that satisfies

$$g(l, \underline{l}) = -1 \ , \quad g(l, v) = 0 \quad \forall v \in T\mathfrak{H}_{u\underline{0}} \ ;$$

similarly $\underline{l}$ can be defined on $\mathfrak{H}^+$. Let $\mathfrak{H}^{+u}$ be the geodesic congruence generated by l initiated from $\mathfrak{H}_{u\underline{0}}$; and $\mathfrak{H}^{-\underline{u}}$ the geodesic congruence generated by $\underline{l}$ initiated from $\mathfrak{H}_{0\underline{u}}$. These two congruences are well-defined on a sufficiently small neighborhood O of $\mathfrak{H}_0$. Define the function u such that its level surfaces are $\mathfrak{H}^{+u}$ and $\underline{u}$ such that its level surfaces are $\mathfrak{H}^{-\underline{u}}$. We also write $\mathfrak{H}_{u\underline{u}} := \mathfrak{H}^{+u} \cap \mathfrak{H}^{-\underline{u}}$. By definition $u, \underline{u}$ are both positive in $O \cap \mathfrak{D}$. By construction they are optical functions:

$$g(\nabla u, \nabla u) = g(\nabla \underline{u}, \nabla \underline{u}) = 0 \ . \tag{2.28}$$

Define in O the function

$$\Omega := g(\nabla u, \nabla \underline{u}) \ , \tag{2.29}$$

and observe that $\Omega|_{\mathfrak{H}^+ \cup \mathfrak{H}^-} = 1$. Define the vector fields $L, \underline{L}$ as

$$L^a = g^{ab} \nabla_b u \ , \quad \underline{L}^a = g^{ab} \nabla_b \underline{u} \tag{2.30}$$

so L agrees with l on $\mathfrak{H}^\pm$, and $\underline{L}$ agrees with $-\underline{l}$ on $\mathfrak{H}^\pm$. They satisfy

$$g(L, L) = g(\underline{L}, \underline{L}) = 0 \ , \quad g(L, \underline{L}) = \Omega \ .$$

We will also write O_ϵ for the neighborhood

$$O_\epsilon := \{x \in O : |u|, |\underline{u}| < \epsilon\} \ . \tag{2.31}$$

By continuity, there exists a small ϵ_0 such that O_{ϵ_0} is compactly included in O, and such that $\Omega > \frac{1}{2}$ on O_{ϵ_0}.

A null frame can be easily completed on $\mathfrak{H}^\pm$ by the following definitions of $m, \bar{m}$. Choose $m, \bar{m}$ as complex vectors on $T\mathfrak{H}_0$ (we will not be able to define them on the entirety of $\mathfrak{H}_0$, as $\mathbb{S}^2$ is not parallelizable; but it suffices to consider an open neighborhood of $\mathfrak{H}_0$ at a time). We Lie transport m and $\bar{m}$ along the null generators by l and $\underline{l}$; that is, we require $[m, l] = 0$ on $\mathfrak{H}^+$ and $[m, \underline{l}] = 0$ on $\mathfrak{H}^-$. While Lie transport, in general, will not preserve the inner products $g(m, m)$ and $g(m, \bar{m})$, on $\mathfrak{H}^\pm$ we have that the null expansion θ (or $\underline{\theta}$) and null shear ϑ (or $\underline{\vartheta}$) vanish, which guarantees that $g(m, m) = 0$ and $g(m, \bar{m}) = 1$ when $m, \bar{m}$ are constructed this way.

Notice that by construction $m(u) = 0$ on $\mathfrak{H}^+$ and $\nabla_l(m(\underline{u})) = \nabla_m(l(\underline{u})) = 0$ on $\mathfrak{H}^+$, which by the fact that $m(\underline{u}) = 0$ on $\mathfrak{H}_0$ means $m(\underline{u}) = 0$ on $\mathfrak{H}^+$. Arguing similarly on $\mathfrak{H}^-$ gives us that $m, \bar{m}$ are in $T\mathfrak{H}_{u\underline{0}}$ and $T\mathfrak{H}_{0\underline{u}}$.

The particular advantage of this localized null frame on $\mathfrak{H}^+$ and $\mathfrak{H}^-$ is that, by the requirement $[l, m] = 0$, we have $\nabla_l m = \nabla_m l$, so the Ricci coefficients $\Gamma_{\alpha 14} = \Gamma_{\alpha 41}$. In particular, in view of the calculations in the previous section, this implies that on $\mathfrak{H}^+$ $\Gamma_{124} = 0$, and on $\mathfrak{H}^-$ $\Gamma_{123} = 0$, further simplifying the calculation when our attention is restricted to the surfaces $\mathfrak{H}^{\pm}$.

Chapter 3

A characterization of the Kerr-Newman black holes

In this section, a local characterization of the Kerr-Newman metric is obtained. The heuristic argument behind the characterization lies in the special aligned, Petrov type D algebraic structure of the Kerr-Newman metric. We begin by giving a brief review of the algebraic structure of Kerr-Newman metric, in particular the notion of principal null directions.

First consider the case of a two-form X_{ab}. It can be considered as an anti-symmetric map on the space of vectors. Because the metric on the tangent space is Minkowskian, we can ask for its eigenvectors. One immediately sees that if r^a is an eigenvector of X_{ab},

$$\lambda r^a r_a = X_{ab} r^b r^a = 0$$

the last equality by the antisymmetry. Hence either the eigenvalue $\lambda = 0$ or r^a is a null vector. In the latter case, r^a is said to be a principal null vector of X_{ab}. By the classification theorem (a fact made immediately obvious in the spinor decomposition, see e.g. [30]), any non-zero two-form must admit either one repeated principal null direction, or two distinct principal null directions. In the former case the two-form is said to be null, and it satisfies $\mathcal{X}^2 = 0$ where $\mathcal{X}_{ab}$ is the anti-self-dual part of X_{ab}. Note that the eigenvalue equation can be re-written in the following form:

$$r_{[c}X_{a]b}r^b = 0 \ .$$

Now consider the case of an algebraically Weyl field. It can be viewed as a trace-free symmetric map from two-forms to two-forms. Therefore we can also consider its eigenvectors (where the "vectors" now are two-forms). But recalling that the two-forms also admit classification by principal null vectors, we observe that this classification can be immediately passed upward to the level of Weyl fields. In particular, we say that a vector r^a is a principal null vector of an algebraically Weyl field C_{abcd} if

$$r^b r^c r_{[e}C_{a]bc[d}r_{f]} = 0$$

in direct analogue to the spin-1/two-form case. The Petrov classification (see [45, 39, 30]) gives the possible multiplicities of principal null directions; again, this fact is most obvious in the spinor language. We will not discuss all of the Petrov types here, it suffices to say that up to multiplicity, there must be exactly four principal null directions for a Weyl field (unless the field vanishes completely). An algebraically Weyl field is said to be *Petrov type D* if it admits two distinct principal null directions each with multiplicity 2.

The Kerr-Newman space-time features a triple alignment of principal null directions. On the Kerr-Newman space-time, the two two-forms, Ernst F_{ab} and Maxwell H_{ab}, are naturally defined. Both of the two-forms are everywhere non-null and each admits two distinct principal null directions. Furthermore, the principal null directions for the Ernst and Maxwell two-forms agree everywhere. In addition, the Weyl curvature is everywhere Petrov type D, and the principal null directions for the Ernst and Maxwell two-forms are also each repeated principal null directions for the Weyl curvature. This is the starting point of the characterization given in this chapter.

Historically much work has gone into the study of non-null solutions to the Einstein-Maxwell equations. A comprehensive study of this subject goes under the name of Rainich theory, whereby the Einstein-Maxwell equations are reduced to a set of algebraic conditions and the Rainich differential equations. An interesting recent work of J.J. Ferrando and J.A. Sáez [10] starts from Rainich theory and shows that if one restricts to the class of non-null solutions to the Einstein-Maxwell system which are Petrov type D and has alignment of the Weyl and Maxwell principal null directions (stationarity is not assumed in their work), the solutions can be classified completely base on purely algebraic conditions, and not differential ones. One can consider the result given in this chapter a special case of the result of Ferrando and Sáez, in which the metric becomes completely integrable. The chief difference, of course, is that here a triple alignment is assumed, which is available only because we focus on the stationary class.

Heuristically, we follow the approach of Mars [22] in the construction of our characterization. The alignment of principal null vectors for two two-forms X_{ab}, Y_{ab} can be expressed as

$$\mathcal{X}_{ab} \propto \mathcal{Y}_{ab}$$

where the scalar of proportionality is a complex number. Now, one can observe through direct computation that the symmetric spinor product of two two-forms $(\mathcal{X}\tilde{\otimes}\mathcal{Y})_{abcd}$ is a Weyl field with principal null directions the union of the principal null directions of X_{ab} and Y_{ab}, thus if X_{ab} is non-null, $(\mathcal{X}\tilde{\otimes}\mathcal{X})_{abcd}$ must be type D. So we can write the alignment condition of the principal null directions of the Weyl curvature as

$$\mathcal{C}_{abcd} \propto (\mathcal{X}\tilde{\otimes}\mathcal{X})_{abcd} \ .$$

Just a proportionality is, however, not enough. We need to know the exact factor to characterize Kerr-Newman metric[1].

The necessity of a scalar function that desribed the proportionality factors[2] can be interpreted as the following. It has been shown in Chapter 2 that the Ernst two-form F_{ab} satisfies a Maxwell equation with source which depends on the stress-energy tensor of the Maxwell field H_{ab} and its own vector potential t_a. In otherwords, the Ernst two-form behaves like an electro-magnetic wave minimally coupled to the background geometry but also driven through interaction with the free Maxwell field. The Weyl curvature, on the other hand, can be seen to satisfy a divergence-curl system (a spin-2 wave equation) by examining the second Bianchi identities. Thus the Weyl field is also seen to be a spin-2 wave that is driven by the Maxwell field. By the complexification procedure, the system is essentially invariant under a $U(1)$ gauge action. So we have the following heuristic analysis of degrees of freedom for the system:

1. The principal null directions of the three waves. This we fix to be aligned by assumption.

2. The frequency of the three waves. This we fix to be identical by the assumption that all three waves are fixed by the Killing action of t^a.

3. The amplitude of the three waves. This should not require fixing, as one may expect the ratio of the amplitudes should give us the charge/mass and angular-momentum/mass ratios of the space-time, and so should be free. The exact amplitude of the Ernst two form and the amplitude of the Maxwell form are, of

[1]Strictly speaking, this statement is false. As Mars indicated in [23], just the proportionality expression suffices, with assumption of stationary asymptotic flatness, for showing a local isometry from a given solution of the Einstein vacuum equations to the Kerr metric. That the proportionality factor is necessary in this work is due to more subtle reasons, the two main ones being that (1) we require a purely local characterization of the Kerr-Newman metric whereas the result in [23] requires non-local information from spatial infinity; and (2) it is impossible to work analytically with an algebraic alignment condition without the proportionality factor. Both of these reasons are motivated by our analytic approach to the conditional uniqueness theorem for Kerr-Newman metrics; see Chapter 4. It will be an interesting further project, separate from the goals of the current work, to see if the characterization derived herein admits a generalization à la Mars [23].

[2]Beyond this heuristic justification, see also Remark 3.3.8 below for another reason why the proportionality factors are useful.

course, related by the Komar mass formulae to the asymptotic mass and charge of the stationary space-time.

4. The phase difference of the three waves.

Hence to uniquely describe Kerr-Newman space-time, we heuristically expect to need to know certain renormalizable (see Theorem 3.5.1) factors of proportionality that describes, essentially, the phase difference between the three waves (see Theorem 3.2.1 for precise description). Once all these degrees of freedom are fixed, we can, in principle, directly integrate the Einstein equations and show that the metric must be Kerr-Newman.

The method employed by M. Mars and the present chapter bears much similarity to the work of R. Debever, N. Kamran, and R.G. McLenaghan [9]. In that work, the authors assumed (i) the space-time is of Petrov type D, (ii) the principal null directions of the Maxwell tensor align (nonsingularly) with that of the Weyl tensor, (iii) a technical hypothesis to allow the use of the generalized Goldberg-Sachs theorem (see Chapter 7 in [39] for example and references), and integrated the Newman-Penrose variables to arrive at explicit local forms of the metric in terms of several free constants and several unknown functions. In view of the work of Debever et al., the assumptions taken in this chapter merely guarantees that their hypotheses (i) and (ii) hold, and that (iii) becomes ancillary to a stronger condition derived herein that circumvents the Goldberg-Sachs theorem as well as reduces the amount of freedom in the local form of the metric.

We should also mention the work of D. Bini et al. [1] on a different generalization of the result of Mars, in which they keep the same definition of the Mars-Simon tensor, while modifying the definition of the Simon three-tensor [36, 37] with a source term that corresponds to the stress-energy tensor associated to the electromagnetic field. They were then able to show that the vanishing of the modified Simon tensor implies also the alignment of principal null directions. In the present work, we absorb the source term into the Mars-Simon tensor itself using only space-time quantities by sacrificing a need for an auxiliary two-form, thus we are able to argue in much of the same way as Mars [22] an explicit computation for the metric expressed in local coördinates, thereby giving a characterization of the Kerr-Newman space-time.

3.1 The basic assumptions on the space-time

Take a space-time $(\mathcal{M}, g_{ab})$ and a Maxwell two-form H_{ab} on $\mathcal{M}$ such that they solve the Einstein-Maxwell equations (see Section 1.1 for definitions). Assume the solution satisfies the following assumptions

(A1) $\mathcal{M}$ is simply-connected.

(A2) $(\mathcal{M}, g_{ab})$ admits a non-trivial smooth Killing vector field t^a, and the Maxwell field H_{ab} inherits the Killing symmetry, i.e. its Lie derivative $\mathcal{L}_t H_{ab} = 0$.

In the sequel a local and a global[3] version of the result will be stated. For the local theorem, it is necessary to also assume

(L) the Killing vector field t^a is time-like somewhere on the space-time $(\mathcal{M}, g_{ab})$, and H_{ab} is non-null on $\mathcal{M}$. (In other words $\mathcal{H}_{ab}\mathcal{H}^{ab} \neq 0$ everywhere on $\mathcal{M}$.)

And for the global result, it is necessary to assume

(G) that $(\mathcal{M}, g_{ab})$ contains a stationary asymptotically flat end $\mathcal{M}^\infty$ where t^a tends to a time translation at infinity, with the Komar mass M of t^a non-zero in $\mathcal{M}^\infty$. In addition the total charge $q = \sqrt{q_E^2 + q_B^2}$ of the Maxwell field, where q_E and q_B denote the electric and magnetic charges, is non-zero in $\mathcal{M}^\infty$.

[3]The world "global" here is used to mean "non-local". More precisely, in the theorems proven below, we will not be able to obtain a global isometry from a given stationary, algebraically aligned solution $(\mathcal{M}, g_{ab}, H_{ab})$ to the Einstein-Maxwell system to the Kerr-Newman space-time. Rather, the word "global" here means that we extract global (or non-local) information from asymptotic flatness to relax certain assumptions. The end result is still a local isometry.

Remark 3.1.1. *Here the definition of stationary asymptotically flat end is quickly recalled:* $\mathcal{M}^\infty$ *is an open submanifold of* $\mathcal{M}$ *diffeomorphic to* $(t_0, t_1) \times (\mathbb{R}^3 \backslash \bar{B}(R))$ *with the metric stationary in the* t *variable,* $\partial_t g_{ab} = 0$*, and satisfying the decay condition*

$$|g_{ab} - \eta_{ab}| + |r\partial g_{ab}| \leq Cr^{-1}$$

for some constant C*;* r *is the radial coördinate on* $\mathbb{R}^3$ *and* η *is the Minkowski metric. In addition, a decay condition for the Maxwell field is required:*

$$|H_{ab}| + |r\partial H_{ab}| \leq C' r^{-2}$$

for some constant C'*.*

Observe that under assumption (G), by the asymptotic flatness, the complex scalar Ξ defined in Section 2.2 has a unique limit at spatial infinity. So a natural choice of normalization is to set $\Xi \to 0$ as $r \to \infty$.

3.2 The tensor characterization of Kerr-Newman space-time; statement of the main theorems

We first state the main result of this chapter, which establishes a purely local characterization of the Kerr-Newman metric. This formulation is comparable to that of Theorem 1 in [23]. The conditions given below on the constants C_2 and C_4 are analogous to the conditions for the constants l and c in the aforementioned theorem.

Theorem 3.2.1 (Main Local Theorem)**.** *Let* $(\mathcal{M}, g_{ab}, H_{ab})$ *solve the Einstein-Maxwell system. Assuming (A1), (A2) and (L), and assuming that there exists a complex scalar* P*, a normalization for* Ξ*, and a nonzero complex constant* C_1 *such that*

1. $P^{-4} = -C_1^2 \mathcal{H}_{ab}\mathcal{H}^{ab}$

2. $\mathcal{F}_{ab} = 4\bar{\Xi}\mathcal{H}_{ab}$

3. $\mathcal{C}_{abcd} = 3P(\mathcal{F}\tilde{\otimes}\mathcal{H})_{abcd}$

then we can conclude

1. there exists a complex constant C_2 *such that* $P^{-1} - 2\Xi = C_2$*;*

2. there exists a real constant C_4 *such that* $t_a t^a + 4|\Xi|^2 = C_4$*.*

If C_2 *further satisfies that* $C_1\bar{C}_2$ *is real, and that* C_4 *is such that* $|C_2|^2 - C_4 = 1$*, then we also have*

3. $\mathfrak{A} = |C_1|^2 P\bar{P}(\Im C_1 \nabla P)^2 + (\Im C_1 P)^2$ *is a non-negative real constant on the manifold*[4]*,*

4. and $(\mathcal{M}, g_{ab})$ *is locally isometric to a Kerr-Newman space-time of total charge* $|C_1|$*, angular momentum* $\sqrt{\mathfrak{A}}C_1\bar{C}_2$*, and mass* $C_1\bar{C}_2$*.*

The local theorem yields, via a simple argument, the following characterization of the Kerr-Newman metric among stationary asymptotically flat solutions to the Einstein-Maxwell system.

Corollary 3.2.2 (Main Global Result)**.** *Let* $(\mathcal{M}, g_{ab}, H_{ab})$ *solve the Einstein-Maxwell system. We assume (A1), (A2) and (G), and let* q_E*,* q_B*, and* M *be the electric charge, magnetic charge, and Komar mass of the space-time at one asymptotic end. We choose the normalization for* Ξ *such that it vanishes at spatial infinity. If we assume there exists a complex function* P *defined wherever* $\mathcal{H}^2 \neq 0$ *such that*

1. $P^{-4} = -(q_E + iq_B)^2 \mathcal{H}_{ab}\mathcal{H}^{ab}$ *when* $\mathcal{H}^2 \neq 0$

2. $\mathcal{F}_{ab} = (4\bar{\Xi} - \frac{2M}{q_E + iq_B})\mathcal{H}_{ab}$ *everywhere*

[4] $\Im$ will be used to denote the imaginary part of an expression. Notice that $\mathfrak{A}$ is well-defined even though C_1 can be replaced by $-C_1$. One should observe the freedom to replace C_1 by $-C_1$ also manifests in the remainder of this chapter; it shall not be further remarked upon.

3. $\mathcal{C}_{abcd} = 3P(\mathcal{F}\tilde{\otimes}\mathcal{H})_{abcd}$ *when* P *is defined*

then we can conclude that

1. $\mathcal{H}^2$ *is non-vanishing globally,*

2. $\mathfrak{A} = (q_E^2 + q_B^2)P\bar{P}(\Im(q_E + iq_B)\nabla P)^2 + (\Im(q_E + iq_B)P)^2$ *is a non-negative real constant on the manifold,*

3. *and* $(\mathcal{M}, g_{ab})$ *is everywhere locally isometric to a Kerr-Newman space-time of total charge* $q = \sqrt{q_E^2 + q_B^2}$*, angular momentum* $\sqrt{\mathfrak{A}}M$*, and mass* M*.*

Remark 3.2.3. *If one explicitly computes the relevant scalars for the Kerr-Newman metric in the Boyer-Lindquist coördinates (1.3) and (1.4), one sees that by taking* $H_{ab} = (dA)_{ab}$*,*

$$\mathcal{H}^2 = -\frac{q^2}{(r + ia\cos\theta)^4}$$

and

$$P = \frac{r + ia\cos\theta}{q} \ .$$

The Kerr-Newman metric is inherently Petrov type D with the triple alignment discussed in the beginning of this chapter. The Weyl and Maxwell scalars obtained from the null tetrad decomposition described later in (3.3) can also be calculated

$$\Upsilon_0 = \frac{q}{2(r + ia\cos\theta)^2} \ , \quad \Psi_0 = -\frac{q^2 - Mr + iaM\cos\theta}{(r - i\cos\theta)(r + i\cos\theta)^3} \ .$$

The Υ_0 *component of the Ernst two form can also be written down as*

$$\Upsilon_0^{(\mathcal{F})} = \frac{q^2 - Mr + iaM\cos\theta}{(r - ia\cos\theta)(r + ia\cos\theta)^2} \ .$$

For ease of notation, we write the complex scalar P, the complex anti-self-dual form $\mathcal{B}_{ab}$, and the complex anti-self-dual Weyl field $\mathcal{Q}_{abcd}$ for the following expressions

$$P^4 := -\frac{1}{C_1^2 \mathcal{H}_{ab}\mathcal{H}^{ab}} \tag{3.1a}$$

$$\mathcal{B}_{ab} := \mathcal{F}_{ab} + (2\bar{C}_3 - 4\bar{\Xi})\mathcal{H}_{ab} \tag{3.1b}$$

$$\mathcal{Q}_{abcd} := \mathcal{C}_{abcd} - 3P(\mathcal{F}\tilde{\otimes}\mathcal{H})_{abcd} \tag{3.1c}$$

By an abuse of language, in the sequel, the statement "$\mathcal{B}_{ab} = 0$" will be understood to mean the alignment condition (2) in Theorem 3.2.1 when we work under assumption (L), or the alignment condition (2) in Corollary 3.2.2 when we work under assumption (G), with suitably defined constants and normalizations. Similarly, the statement "$\mathcal{Q}_{abcd} = 0$" will be taken to mean the existence of a suitable function P such that the appropriate alignment condition (3) is satisfied under suitable conditions.

3.3 Proof of the main local theorem

Throughout this section we assume the statements (A1), (A2) and (L). The arguments in this section, except for Lemma 3.3.1 and Proposition 3.3.2, closely mirror the arguments given in [22], with several technical changes to allow the application to electrovac space-times. Using the precise statement of Theorem 3.2.1, C_3 should be taken to be 0 in this section. We keep the notation C_3 to make explicit the applicability of the computations in the global case.

We start first with some consequences of assumption (L)

Lemma 3.3.1. *If $\mathcal{B}_{ab}$ vanishes identically on $\mathcal{M}$, then we have that*

1. *$\mathcal{F}_{ab}\mathcal{F}^{ab}$ only vanishes on sets of co-dimension ≥ 1,*

2. *$\mathcal{F}_{ab}\mathcal{F}^{ab} = 0 \implies \mathcal{F}_{ab} = 0$,*

3. *The Killing vector field t^a is non-null on a dense subset of $\mathcal{M}$.*

Proof. Squaring the alignment condition implied by the vanishing of $\mathcal{B}_{ab}$ gives

$$\mathcal{F}^2 = (4\bar{\Xi} - 2\bar{C}_3)^2 \mathcal{H}^2 \ .$$

By assumption (L), if the left-hand side vanishes, then $4\Xi - 2C_3 = 0$, and using the alignment condition again, we have $\mathcal{F}_{ab} = 0$. This proves claim (2).

Suppose $\mathcal{F}_{ab}$ vanishes on some small open set δ, then necessarily $\nabla_a t_b = 0$ on δ. Furthermore, we have that Ξ must be locally constant as shown above, and thus $\nabla_a \Xi = \mathcal{H}_{ba} t^b = 0$. But

$$\nabla_a \Xi \nabla^a \Xi = \mathcal{H}_{ba} \mathcal{H}^{ca} t_c t^b = \frac{1}{4} \mathcal{H}_{ab} \mathcal{H}^{ab} t^c t_c = 0$$

and since the Maxwell field is non-null, we have that t^a must be a parallel null vector in δ. If t^a is not the zero vector, however, we must have t^a being an eigenvector, and hence a principal null direction, of $\mathcal{H}_{ab}$, with eigenvalue zero: this contradicts the fact that $\mathcal{H}_{ab}$ is non-null. If $t^a = 0$ on a small neighborhood δ, however, t^a must vanish everywhere on $\mathcal{M}$ since it is Killing, contradicting assumption (A2). This proves assertion (1).

Lastly, assume that $t^2 = 0$ on some small open set δ, which implies $\nabla_a t^2 = 0$ and $\Box_g t^2 = 0$ on the neighborhood. Using (2.20), we deduce

$$^*F_{mx} t^{x*} F_{ny} t^y = \frac{1}{2} F^2 t_m t_n \ .$$

Taking the trace in m, n, we have

$$^*F_{mx}{}^*F^{my} t_y t^x = 0 \ .$$

Using the fact that

$$F_{mx}F^{my}t^x t_y = \nabla_m t^2 \nabla^m t^2 = 0 \ , \qquad \mathcal{F}_{ac}\bar{\mathcal{F}}_b{}^c = \frac{1}{4}(F_{ac}F_b{}^c + {}^*F_{ac}{}^*F_b{}^c)$$

we have

$$\mathcal{F}_{ac}\bar{\mathcal{F}}_b{}^c t^a t^b = 0 \ .$$

Now, since $\mathcal{B}_{ab} = 0$, this implies that

$$|2C_3 - 4\Xi|^2 T_{ab}t^a t^b = 0$$

on the open set δ. If the first factor is identically zero in an open subset $\delta' \subset \delta$, then Ξ is locally constant and arguing the same way as before we get a contradiction. Therefore we can assume, without loss of generality, that $T_{ab}t^a t^b = 0$ on our open set δ. Now consider the identity

$$0 = \Box_g t^2 = \nabla^b(t^a F_{ba}) = \frac{1}{2}F^{ba}F_{ba} - 2R_{ab}t^a t^b \ .$$

The last term vanishes by the assumption, and implies that $F^{ba}F_{ba} = 0$; thus ${}^*F_{mx}t^x = 0$. Therefore

$$\nabla_a t^2 = t^b F_{ab} = 2t^b \mathcal{F}_{ab}$$

in δ, and hence

$$0 = \Box_g t^2 = \mathcal{F}_{ab}\mathcal{F}^{ab} - 2R_{ab}t^a t^b$$

and so $\mathcal{F}_{ab}\mathcal{F}^{ab} = 0$ identically on δ, which we have just shown is impossible. Assertion (3) then follows. $\Box$

We can then prove claim (1) in Theorem 3.2.1:

Proposition 3.3.2. *If $\mathcal{B}_{ab}$ and $\mathcal{Q}_{abcd}$ both vanish on $\mathcal{M}$, then $P^{-1} - 2\Xi$ is constant.*

Proof. We start by calculating $\mathcal{H}^{ab}\nabla_c\mathcal{B}_{ab}$. Using (2.18),

$$\begin{aligned}\mathcal{H}^{ab}\nabla_c\mathcal{F}_{ab} &= 2[\mathcal{Q}_{dcab} + 3P(\mathcal{F}\tilde{\otimes}\mathcal{H})_{dcab}]t^d\mathcal{H}^{ab} \\ &\quad + \frac{1}{2}(T_{ad}\mathcal{H}^a{}_c + T_{bc}\mathcal{H}_d{}^b - T_{ac}\mathcal{H}^a{}_d - T_{bd}\mathcal{H}_c{}^b)t^d \\ &\quad + i(T_d{}^{e*}\mathcal{H}_{ec} + T_c{}^{f*}\mathcal{H}_{df})t^d \\ &= 2[\mathcal{Q}_{dcab} + 3P(\mathcal{F}\tilde{\otimes}\mathcal{H})_{dcab}]t^d\mathcal{H}^{ab} + 2(T_{ad}\mathcal{H}^a{}_c + T_{bc}\mathcal{H}_d{}^b)t^d \\ &= 2\mathcal{Q}_{dcab}\mathcal{H}^{ab}t^d + P(3\mathcal{F}_{dc}\mathcal{H}_{ab}\mathcal{H}^{ab} + \mathcal{H}_{dc}\mathcal{F}_{ab}\mathcal{H}^{ab})t^d \\ &\quad + 8(\mathcal{H}_{af}\bar{\mathcal{H}}_d{}^f\mathcal{H}^a{}_c + \mathcal{H}_{bf}\bar{\mathcal{H}}_c{}^f\mathcal{H}_d{}^b)t^d \\ &= 2\mathcal{Q}_{dcab}\mathcal{H}^{ab}t^d + P(3[C_1^{-1}\mathcal{B}_{dc} - (2\bar{C}_3 - 4\bar{\Xi})\mathcal{H}_{dc}]\mathcal{H}_{ab}\mathcal{H}^{ab} \\ &\quad + \mathcal{H}_{dc}[C_1^{-1}\mathcal{B}_{ab} - (2\bar{C}_3 - 4\bar{\Xi})\mathcal{H}_{ab}]\mathcal{H}^{ab})t^d + 4\mathcal{H}_{ab}\mathcal{H}^{ab}\bar{\mathcal{H}}_{dc}t^d\end{aligned}$$

where we used (2.4d) and (3.1b) in the last equality. Using (3.1a), we simplify to

$$\begin{aligned}\mathcal{H}^{ab}\nabla_c\mathcal{F}_{ab} = 2\mathcal{Q}_{dcab}\mathcal{H}^{ab}t^d - \frac{3}{C_1^3P^3}\mathcal{B}_{dc}t^d + \frac{4}{C_1^2P^3}(2\bar{C}_3 - 4\bar{\Xi})\mathcal{H}_{dc}t^d \\ + C_1^{-1}P\mathcal{H}_{dc}\mathcal{B}_{ab}\mathcal{H}^{ab}t^d - \frac{4}{C_1^2P^4}\bar{\mathcal{H}}_{dc}t^d\end{aligned}$$

Applying the condition $\mathcal{Q}_{abcd} = 0$ and $\mathcal{B}_{ab} = 0$ and (2.12), we have

$$\mathcal{H}^{ab}\nabla_c\mathcal{F}_{ab} = \frac{4}{C_1^2P^3}(2\bar{C}_3 - 4\bar{\Xi})\nabla_c\Xi - \frac{4}{C_1^2P^4}\nabla_c\bar{\Xi}$$

On the other hand, we can calculate

$$\mathcal{H}^{ab}\nabla_c\left[(2\bar{C}_3 - 4\bar{\Xi})\mathcal{H}_{ab}\right] = -4\mathcal{H}^{ab}\mathcal{H}_{ab}\nabla_c\bar{\Xi} + \frac{1}{2}(2\bar{C}_3 - 4\bar{\Xi})\nabla_c(\mathcal{H}_{ab}\mathcal{H}^{ab})$$

So putting them altogether we have

$$0 = \mathcal{H}^{ab}\nabla_c\mathcal{B}_{ab} = \frac{4}{C_1P^3}(\bar{C}_3 - 2\bar{\Xi})(2\nabla_c\Xi - \nabla_c\frac{1}{P})$$

By the arguments used in the proof of Lemma 3.3.1, Ξ is not locally constant and so $C_3 \neq 2\Xi$ densely. The above expression (and continuity) then shows that $2\Xi - \frac{1}{P}$ is constant. □

In what follows I'll write $C_2 = P^{-1} - 2\Xi + C_3$.

Remark 3.3.3. *In the global case (where we assume (G) instead of (L)), the decay condition given by asymptotic flatness shows that 2Ξ and $1/P$ both vanish at spatial infinity, and so $C_2 = C_3 = M/(q_E - iq_B)$ everywhere.*

The next proposition demonstrates assertion (2) in Theorem 3.2.1.

Proposition 3.3.4. *Assuming the vanishing of $\mathcal{B}_{ab}$ and $\mathcal{Q}_{abcd}$, we have the following identities*

$$t^2 = -\left|\frac{1}{P} - C_2\right|^2 + C_4 \tag{3.2a}$$

$$(\nabla P)^2 = -\frac{t^2}{C_1^2} \tag{3.2b}$$

$$C_1 \Box_g P = -\frac{2}{C_1 \bar{C}_1 P \bar{P}} \left(\bar{C}_1 C_2 - (|C_2|^2 - C_4)\bar{C}_1 \bar{P}\right) \tag{3.2c}$$

where C_4 is a real-valued constant.

Proof. We can calculate

$$\nabla_a t^2 = 2t^b \nabla_a t_b = -F_{ba} t^b = -2\Re[\mathcal{F}_{ba} t^b]$$

The vanishing of $\mathcal{B}_{ab}$ and Proposition 3.3.2 together imply

$$\nabla_a t^2 = -4\Re[(2\bar{\Xi} - \bar{C}_3)\mathcal{H}_{ba} t^b] = -2\Re[(\frac{1}{\bar{P}} - \bar{C}_2)\nabla_a \frac{1}{P}] = -\nabla_a \left|\frac{1}{P} - C_2\right|^2$$

The first claim follows as $\mathcal{M}$ is simply connected. Next, from Proposition 3.3.2 we get

$$\nabla_a P = \nabla_a \frac{1}{2\Xi + C_2 - C_3} = -\frac{2\nabla_a \Xi}{(2\Xi + C_2 - C_3)^2} = -2P^2 \mathcal{H}_{ba} t^b$$

So

$$\nabla_a P \nabla^a P = 4P^4 \mathcal{H}_{ba} t^b \mathcal{H}^{ca} t_c = P^4 \mathcal{H}^2 t^2 = -\frac{t^2}{C_1^2}$$

where we used (2.4d) and the definition for P. We can also calculate directly the D'Alembertian

$$\begin{aligned}
\Box_g P &= -2\nabla^a (P^2 \mathcal{H}_{ba} t^b) \\
&= -2\mathcal{H}_{ba}(2P\nabla^a P t^b + \frac{1}{2} P^2 F^{ab}) \\
&= 2\mathcal{H}_{ba}(4P^3 \mathcal{H}^{ca} t_c t^b + \frac{1}{2} P^2 \mathcal{F}^{ba}) \\
&= 2P^3 \mathcal{H}^2 t^2 + 2P^2 (\frac{1}{\bar{P}} - \bar{C}_2) \mathcal{H}^2 \\
&= 2P^2 \mathcal{H}^2 \left[P \left(-\left| \frac{1}{P} - C_2 \right|^2 + C_4 \right) + \frac{1}{\bar{P}} - \bar{C}_2 \right] \\
&= 2P^2 \mathcal{H}^2 \left[\left(\frac{1}{\bar{P}} - \bar{C}_2 \right) \left(1 - P \left(\frac{1}{P} - C_2 \right) \right) + C_4 P \right] \\
&= \frac{2}{C_1^2 P} \left(C_2 (\frac{1}{\bar{P}} - \bar{C}_2) + C_4 \right)
\end{aligned}$$

from which the third identity follows by simple algebraic manipulations. □

Remark 3.3.5. *If we further impose the condition that $C_1 \bar{C}_2$ is real, then the imaginary part of the third identity becomes*

$$\Im(\Box_g C_1 P) = \frac{2(|C_2|^2 - C_4)}{|C_1 P|^2} \Im(C_1 P)$$

which will be useful later. In the global case, we can again match the data at spatial infinity to see that $C_4 = |C_2|^2 - 1 = M^2/q^2 - 1$ (the condition relating C_2 and C_4 in

Theorem 3.2.1 is directly satisfied); the third identity then reads:

$$(q_E + iq_B)\Box_g P = -\frac{2}{q^2 P\bar{P}}\left(M - (q_E - iq_B)\bar{P}\right)$$

An immediate consequence of the above proposition is that $(\nabla C_1 P)^2$ is real. Writing the complex quantity $C_1 P = y + iz$, where y and z are real-valued, we see that this implies

$$\nabla^a y \nabla_a z = 0$$

Furthermore, by Lemma 3.3.1, we have that, with the possible exception on sets of co-dimension ≥ 1, $t^2 \neq 0$. This leads to the useful observation that, with the possible exception on those points, $(\nabla y)^2$ and $(\nabla z)^2$ cannot simultaneously vanish, and in particular $\nabla_a y$ and $\nabla_a z$ are not simultaneously null, and thus rule out the case where the two are aligned. We summarize in the following

Corollary 3.3.6. *Letting $C_1 P = y + iz$, we know that on any open set*

1. *P is not locally constant*
2. *$\nabla_a y$ and $\nabla_a z$ are mutually orthogonal*
3. *$\nabla_a y$ and $\nabla_a z$ cannot be both null*
4. *$\nabla_a y$ and $\nabla_a z$ cannot be parallel*

Replacing $C_1 P$ by $y + iz$, and imposing the condition $C_1 \bar{C}_2$ is real, we can also rewrite

$$t^2 = -\frac{C_1\bar{C}_1 - 2C_1\bar{C}_2 y}{y^2 + z^2} - |C_2|^2 + C_4 \ .$$

Since $\mathcal{H}_{ab}$ is an anti-self-dual two form with non-vanishing norm, it has two distinct principal null directions, which we denote by $\underline{l}^a$ and l^a, with the normalization $g_{ab}\underline{l}^a l^b = -1$. The alignment of $\mathcal{H}_{ab}$ with $\mathcal{F}_{ab}$ (via vanishing of $\mathcal{B}_{ab}$) allows the following expressions

$$\mathcal{H}_{ab} = \frac{1}{2C_1P^2}(\underline{l}_a l_b - l_a\underline{l}_b + i\epsilon_{abcd}\underline{l}^c l^d)$$
$$\mathcal{F}_{ab} = \frac{\frac{1}{\bar{P}} - \bar{C}_2}{C_1P^2}(\underline{l}_a l_b - l_a\underline{l}_b + i\epsilon_{abcd}\underline{l}^c l^d)$$

By the assumption $\mathcal{Q}_{abcd} = 0$, the principal null directions of $\mathcal{H}_{ab}$ are repeated null directions of the anti-self-dual Weyl tensor, and thus the space-time is algebraically special (Type D). On a local neighborhood, we can take $m, \bar{m}$ complex smooth vector fields to complete the null tetrad $\{m, \bar{m}, \underline{l}, l\}$ (see Section 2.4), and in the tetrad (spinor) formalism, the only non-zero Weyl scalar is

$$\Psi := \Psi_0 = W(\bar{m}, \underline{l}, m, l) = -\frac{1}{C_1^2P^3}\left(\frac{1}{\bar{P}} - \bar{C}_2\right) \tag{3.3a}$$

the only non-zero component of the Maxwell scalars is

$$\Upsilon := \Upsilon_0 = \mathcal{H}_{ab}l^a\underline{l}^b = \frac{1}{2C_1P^2} \tag{3.3b}$$

and the only non-zero component of the Ricci scalars is

$$\Phi := \Phi_0 = T(\underline{l}, l) = T(m, \bar{m}) = \frac{1}{C_1\bar{C}_1P^2\bar{P}^2} \tag{3.3c}$$

Notice the following symmetry relations

$$\bar{\Psi} = \underline{\Psi}\ , \quad \underline{\bar{\Psi}} = \Psi\ , \quad \underline{\bar{\Upsilon}} = -\Upsilon\ , \quad \bar{\Phi} = \underline{\Phi} = \Phi \tag{3.4}$$

Now, from

$$2C_1P^2\mathcal{H}_{ab}t^a = -C_1\nabla_b P$$

we can calculate

$$\nabla_b y = \underline{l}_b\ l_a t^a - l_b\ \underline{l}_a t^a \qquad (\nabla y)^2 = 2 l_a \underline{l}_b t^a t^b \tag{3.5a}$$

$$\nabla_b z = \epsilon_{bacd} t^a \underline{l}^c l^d \qquad (\nabla z)^2 = 2 \underline{l}_a l_b t^a t^b + t^2 \tag{3.5b}$$

So we need expressions for $g(t, \underline{l}), g(t, l)$. From the fact that $\mathcal{L}_t \mathcal{H} = 0$, we have

$$[t, \underline{l}]_a l_b + \underline{l}_a [t, l]_b - [t, l]_a \underline{l}_b - l_a [t, \underline{l}]_b = 0$$

which we can contract against l and $\underline{l}$ (using the fact that $[t, l]_a l^a = \partial_t l^2 = 0$) to arrive at

$$[t, \underline{l}]_a = \underline{l}_a [t, l]_b \underline{l}^b = K_t \underline{l}_a \tag{3.6a}$$

$$[t, l]_a = l_a [t, \underline{l}]_b l^b = -K_t l_a \tag{3.6b}$$

where the function $K_t := [t, l]_b \underline{l}^b$. Now

$$\partial_t (t_b \underline{l}^b) = \mathcal{L}_t (t_b \underline{l}^b) = K_t t_b \underline{l}^b$$

and similarly

$$\partial_t (t_b l^b) = -K_t t_b l^b$$

Lastly, we compute an expression for t by

$$-\frac{\mathcal{H}^{cb} \nabla_b P}{2P^2} = \frac{1}{4} \mathcal{H}^2 t^c = -\frac{t^c}{4 C_1^2 P^4}$$

Therefore, by a direct computation

$$t_c = -(l_a t^a) \underline{l}_c - (\underline{l}_a t^a) l_c - \epsilon_{cabd} (\nabla^a z) \underline{l}^b l^d \tag{3.7}$$

Next is the main lemma of this section, which also gives assertion (3) of Theorem 3.2.1.

Lemma 3.3.7. *Assuming $\mathcal{B}_{ab}$ and $\mathcal{Q}_{abcd}$ vanish, $C_1\bar{C}_2$ is real, and $|C_2|^2 - C_4 = 1$, we have the norms*

$$(\nabla z)^2 = \frac{\mathfrak{A} - z^2}{y^2 + z^2} \tag{3.8a}$$

$$(\nabla y)^2 = \frac{\mathfrak{A} + y^2 + |C_1|^2 - 2C_1\bar{C}_2 y}{y^2 + z^2} \tag{3.8b}$$

where $\mathfrak{A}$ is a non-negative constant with $z^2 \leq \mathfrak{A}$.

Proof. We will use the tetrad formalism of Klainerman-Ionescu (see Section 2.4) extensively in the following computation. By the alignment properties (3.3) and the symmetry properties (3.4), the Maxwell equations simplify to

$$\underline{D}\Upsilon = -2\bar{\underline{\theta}}\Upsilon \qquad D\Upsilon = -2\theta\Upsilon$$

$$-\delta\Upsilon = 2\eta\Upsilon \qquad -\bar{\delta}\Upsilon = 2\bar{\underline{\eta}}\Upsilon$$

from which we arrive at

$$DP = \theta P\ , \quad \underline{D}P = \bar{\underline{\theta}}P\ , \quad \delta P = \eta P\ , \quad \bar{\delta}P = \bar{\underline{\eta}}P \tag{3.9}$$

From the decomposition (3.5) we then have

$$\nabla_a y = -\theta C_1 P \underline{l}_a - \bar{\underline{\theta}} C_1 P l_a \tag{3.10a}$$

$$i\nabla_a z = \eta C_1 P \bar{m}_a + \bar{\underline{\eta}} C_1 P m_a \tag{3.10b}$$

Using the fact that y and z are real, taking complex conjugates on the above equations gives us

$$\theta C_1 P = \bar{\theta}\bar{C}_1\bar{P}\ , \quad \underline{\theta}\bar{C}_1\bar{P} = \bar{\underline{\theta}}C_1 P\ , \quad \eta C_1 P = -\underline{\eta}\bar{C}_1\bar{P} \tag{3.11}$$

The Bianchi equations become

$$0 = \xi(3\Psi + \Phi) \tag{3.12a}$$

$$0 = \vartheta(3\Psi - \Phi) \tag{3.12b}$$

$$-D(\Psi + \frac{1}{2}\Phi) = 3\theta\Psi + \bar{\theta}\Phi \tag{3.12c}$$

$$\bar{\delta}(\Psi - \frac{1}{2}\Phi) = -3\underline{\eta}\Psi + \eta\Phi \tag{3.12d}$$

$$-\delta\Phi = 2(\eta + \underline{\eta})\Phi \tag{3.12e}$$

$$D\Phi = -2(\bar{\theta} + \theta)\Phi \tag{3.12f}$$

Because of the triple alignment given by $\mathcal{B}_{ab} = 0$ and $\mathcal{Q}_{abcd} = 0$, the latter four equations contain essentially the same information as the Maxwell equations. We examine the first two in more detail. Consider the equation $3\Psi \pm \Phi = 0$. This implies

$$3\bar{C}_1\bar{C}_2\bar{P}^2 - 3\bar{C}_1\bar{P} \pm C_1 P = 0$$

or

$$\frac{3C_1\bar{C}_2}{C_1\bar{C}_1}(y^2 - z^2) - (3 \mp 1)y = 0$$
$$\frac{6C_1\bar{C}_2}{C_1\bar{C}_1}yz - (3 \pm 1)z = 0$$

Taking derivatives, we have

$$(\frac{6C_1\bar{C}_2}{C_1\bar{C}_1}y - 3 \pm 1)\nabla y = \frac{6C_1\bar{C}_2}{C_1\bar{C}_1}z\nabla z$$
$$(\frac{6C_1\bar{C}_2}{C_1\bar{C}_1}y - 3 \mp 1)\nabla z = -\frac{6C_1\bar{C}_2}{C_1\bar{C}_1}z\nabla y$$

By the assumption that $C_1\bar{C}_2$ is real, all the coefficients in the above two equations are real. Suppose the equation $3\Psi \pm \Phi = 0$ is satisfied on an open-set, as ∇y and ∇z

cannot be parallel by Corollary 3.3.6, we must have then

$$(\frac{6C_1\bar{C}_2}{C_1\bar{C}_1}y - 3 \pm 1)\nabla y = \frac{6C_1\bar{C}_2}{C_1\bar{C}_1}z\nabla z = 0$$

This implies that y and z are locally constant, which contradicts statement (1) in Corollary 3.3.6. Therefore an equation of the form $3\Psi \pm \Phi = 0$ cannot be satisfied on open sets.

Applying to the Bianchi identities (3.12a,3.12b), we see that $\xi = \vartheta = \underline{\xi} = \underline{\vartheta} = 0$. The relevant null structure equations, simplified with the above observation, are

$$(D + \Gamma_{124})\eta = \theta(\underline{\eta} - \eta) \tag{3.13a}$$

$$-\delta\theta = \zeta\theta + \eta(\theta - \bar{\theta}) \tag{3.13b}$$

Define the quantity $A = C_1\bar{C}_1P\bar{P}(\nabla z)^2$. Equations (3.10b) and (3.11) imply that $(\nabla z)^2 = 2\eta\bar{\eta}C_1\bar{C}_1P\bar{P}$, so

$$\begin{aligned} 0 \leq A &= 2\eta\bar{\eta}C_1^2\bar{C}_1^2P^2\bar{P}^2 \\ &= 2C_1^2\bar{C}_1^2\underline{\eta}\underline{\bar{\eta}}P^2\bar{P}^2 \\ &= -(y^2 + z^2) - (C_1\bar{C}_1 - 2C_1\bar{C}_2 y) - 2\theta\underline{\theta}C_1^2\bar{C}_1^2P^2\bar{P}^2 \end{aligned}$$

where in the last line we used Proposition 3.3.4, Corollary 3.3.6, Equations (3.10a) and (3.11), and the assumption that $|C_2|^2 - C_4 = 1$. By using (3.13a,3.13b) we calculate

$$\begin{aligned} D(\eta\bar{\eta}) &= \theta(\underline{\eta} - \eta)\bar{\eta} + \bar{\theta}(\underline{\bar{\eta}} - \bar{\eta})\eta \\ \delta(\theta\underline{\theta}) &= -\eta(\theta - \bar{\theta})\underline{\theta} - \underline{\eta}(\underline{\theta} - \underline{\bar{\theta}})\theta \end{aligned}$$

Thus, with judicial applications of (3.11)

$$\begin{aligned}
DA &= 2C_1^2\bar{C}_1^2[\theta(\underline{\eta}-\eta)\bar{\eta}+\bar{\theta}(\underline{\bar{\eta}}-\bar{\eta})\eta]P^2\bar{P}^2+4C_1^2\bar{C}_1^2\eta\bar{\eta}(\theta+\bar{\theta})P^2\bar{P}^2 \\
&= 0 \\
\delta A &= -\delta(z^2)+2C_1^2\bar{C}_1^2P^2\bar{P}^2[\eta(\theta-\bar{\theta})\underline{\theta}+\underline{\eta}(\underline{\theta}-\underline{\bar{\theta}})\theta] \\
&\quad -4C_1^2\bar{C}_1^2P^2\bar{P}^2(\eta+\underline{\eta})\theta\underline{\theta} \\
&= -\delta(z^2)
\end{aligned}$$

Since $Dz = \underline{D}z = 0$, we have that the function $A+z^2$ is constant. Define $\mathfrak{A} = A+z^2$. The nonnegativity of A guarantees that $z^2 \leq \mathfrak{A}$, and we have

$$(\nabla z)^2 = \frac{A}{C_1\bar{C}_1P\bar{P}} = \frac{\mathfrak{A}-z^2}{y^2+z^2}$$

and

$$(\nabla y)^2 = (C_1\nabla P)^2+(\nabla z)^2 = \frac{\mathfrak{A}+y^2+C_1\bar{C}_1-2C_1\bar{C}_2y}{(y^2+z^2)}$$

as claimed. □

Remark 3.3.8. *In the proof above we showed that $\xi = \vartheta = \underline{\xi} = \underline{\vartheta} = 0$, a conclusion that in the vacuum case [22] is easily reached by the Goldberg-Sachs theorem. It is worth noting that in general, the alignment of the principal null directions of the Maxwell form and the Weyl tensor is not enough to justify the vanishing of all four of the involved quantities. Indeed, the Kundt-Thompson theorem [39] only guarantees that $\xi\vartheta = \underline{\xi}\underline{\vartheta} = 0$. In our special case the improvement comes from the fact that we not only have alignment of the principal null directions, but also knowledge of the proportionality factor. This allows us to write down the polynomial expression in P and $\bar{P}$ which we used to eliminate the case where only one of ξ and ϑ vanishes.*

In the remainder of this section, we assume that $C_1\bar{C}_2$ is real and $|C_2|^2 - C_4 = 1$ and prove assertion (4) in Theorem 3.2.1. Let us first define two auxillary vector fields. On our space-time, let

$$n^a = (\mathfrak{A} + y^2)t^a + (y^2 + z^2)(t_b l^b \underline{l}^a + t_b \underline{l}^b l^a) \ . \tag{3.14}$$

Define $\mathcal{M}_{\mathfrak{A}} := \{p \in \mathcal{M} | z^2(p) < \mathfrak{A}\}$. On this open subset we can define

$$b^a = \frac{\nabla^a z}{(\nabla z)^2} = \frac{y^2 + z^2}{\mathfrak{A} - z^2} \nabla^a z \ . \tag{3.15}$$

We also define the open subsets $\mathcal{M}_l := \{p \in \mathcal{M} | (t_a l^a)(p) \neq 0\}$ and $\mathcal{M}_{\underline{l}} := \{p \in \mathcal{M} | (t_a \underline{l}^a)(p) \neq 0\}$. Now, notice that in our calulcations above using the tetrad formalism, we have only specified the "direction" of $l, \underline{l}$ and their lengths relative to each other. We still have considerable freedom left to fix the lapse of one of the two vector fields and still retain the use of our formalism. On $\mathcal{M}_{\underline{l}}$, we can choose the vector field $\underline{l}$ such that $t_a \underline{l}^a = 1$ (similarly for l on $\mathcal{M}_l$; the calculations with respect to $\mathcal{M}_l$ are almost identical to that on $\mathcal{M}_{\underline{l}}$, so without loss of generality, we will perform calculations below with respect to $\mathcal{M}_{\underline{l}}$) and the vector field l maintaining $l_a \underline{l}^a = -1$. From (3.5) and Lemma 3.3.7, we have that on $\mathcal{M}_{\underline{l}}$ we can write

$$\nabla_a y = -l_a + \frac{\mathfrak{A} + y^2 + |C_1|^2 - 2C_1 \bar{C}_2 y}{2(y^2 + z^2)} \underline{l}_a = -l_a + U \underline{l}_a \tag{3.16}$$

which implies $l_a t^a = U$, where U is defined on the entirety of $\mathcal{M}$ as

$$U := l_a t^a \underline{l}_b t^b = \frac{1}{2} (\nabla y)^2 \ . \tag{3.17}$$

We consider first a special case when t^a is hypersurface orthogonal.

Proposition 3.3.9. *The following are equivalent:*

1. *z is locally constant on an open subset $\mathcal{U} \subset \mathcal{M}$*

2. *$\mathfrak{A}$ vanishes on $\mathcal{M}$*

3. z vanishes on $\mathcal{M}$

Proof. (2) $\implies$ (3) and (3) $\implies$ (1) follows trivially from Lemma 3.3.7. It thus suffices to show (1) $\implies$ (2). Suppose $\nabla z = 0|_{\mathcal{U}}$. We consider the imaginary part of the third identity in Proposition 3.3.4 à la Remark 3.3.5, which shows that $z = 0|_{\mathcal{U}}$. From Lemma 3.3.7 we have $\mathfrak{A} = 0|_{\mathcal{U}}$, but $\mathfrak{A}$ is a universal constant for the manifold, and thus vanishes identically. □

It is simple to check that $z = 0$ on $\mathcal{M}$ implies $C_1^{-1}\mathcal{H}_{ab}t^a = \nabla_b \frac{1}{C_1 P}$ is real, and so the vanishing of $\mathcal{B}_{ab}$ implies $\mathcal{F}_{ab}t^a = 2(\frac{C_1\bar{C}_1}{C_1 P} - C_1\bar{C}_2)C_1^{-1}\mathcal{H}_{ab}t^a$ is purely real, which by Frobenius' theorem gives that t^a is hypersurface orthogonal.[5]

Proposition 3.3.10. *Assume $\mathfrak{A} = 0$. Then, at any point $p \in \mathcal{M}_{\underline{l}}$ there exists a neighborhood that can be isometrically embedded into the Reissner-Nordström solution.*

This proof closely mirrors that of Proposition 2 in [22].

Proof. We use the same tetrad notation as before. Since $z = 0$, we have $C_1 P = y$ is real, and hence (3.11) implies that $\theta, \underline{\theta}$ are real. Furthermore, $z = 0$ implies via (3.10b) that $\eta = 0 = \underline{\eta}$. The commutator relations then gives

$$[D, \underline{D}] = -\underline{\omega} D + \omega \underline{D}$$

$$[\delta, \bar{\delta}] = \Gamma_{121}\bar{\delta} + \Gamma_{122}\delta$$

which implies that $\{l, \underline{l}\}$ and $\{m, \bar{m}\}$ are integrable. Thus a sufficiently small neighborhood $\mathcal{U}$ can be foliated by 2 mutually orthogonal families of surfaces. We calculate the induced metric on the surface tangent to $\{m, \bar{m}\}$ using the Gauss equation.

[5]As to the question whether t^a can be hypersurface orthogonal without $\nabla z = 0$: in the next part we will consider the case where $\mathfrak{A} \neq 0$ (implying z is nowhere locally constant), and show that in the subset $\mathcal{M}_{\underline{l}} \cap \mathcal{M}_{\mathfrak{A}}$ we have local diffeomorphisms to the Kerr-Newman space-time with non-zero angular momentum, which implies that $\mathfrak{A} = 0$ is characteristic of the Reissner-Nordström metric. Indeed, as we shall see later, the quantity $\mathfrak{A}$ is actually square of the normalized angular momentum of the space-time.

First we calculate the second fundamental form $\chi(X,Y)$ for $X^a = X_1 m^a + X_2 \bar{m}^a$ and $Y^a = Y_1 m^a + Y_2 \bar{m}^a$. By definition $\chi(X,Y)$ is the projection of $\nabla_X Y$ to the normal bundle, so in the tetrad frame, evaluating using the connection coefficients, we have

$$\begin{aligned}
\chi(X,Y)^a &= X_1 Y_1 (\Gamma_{131} l^a + \Gamma_{141} \underline{l}^a) + X_1 Y_2 (\Gamma_{231} l^a + \Gamma_{241} \underline{l}^a) \\
&\qquad + X_2 Y_1 (\Gamma_{132} l^a + \Gamma_{142} \underline{l}^a) + X_2 Y_2 (\Gamma_{232} l^a + \Gamma_{242} \underline{l}^a) \\
&= X_1 Y_1 (\underline{\vartheta} l^a + \vartheta \underline{l}^a) + X_1 Y_2 (\underline{\bar{\theta}} l^a + \bar{\theta} \underline{l}^a) \\
&\qquad + X_2 Y_1 (\underline{\theta} l^a + \theta \underline{l}^a) + X_2 Y_2 (\underline{\bar{\vartheta}} l^a + \bar{\vartheta} \underline{l}^a) \\
&= -\frac{\nabla^a y}{C_1 P} g(X,Y) = -\frac{\nabla^a y}{y} g(X,Y)
\end{aligned}$$

where the last line used the vanishing of ϑ derived in the proof of Lemma 3.3.7 and Equation (3.10a). We recall the Gauss equation

$$R_0(X,Y,Z,W) = R(X,Y,Z,W) - g(\chi(X,W),\chi(Y,Z)) + g(\chi(X,Z),\chi(Y,W))$$

where X,Y,Z,W are spanned by $m, \bar{m}$. Plugging in the explicit form of the Riemann curvature tensor, we can compute by taking $X = Z = m, Y = W = \bar{m}$ the only component of the curvature tensor for the 2-surface

$$\begin{aligned}
R_0(m,\bar{m},m,\bar{m}) &= -\Psi - \bar{\Psi} - \Phi - \frac{(\nabla y)^2}{y^2} \\
&= \frac{C_1 \bar{C}_1}{y^4} - \frac{2 C_1 \bar{C}_2}{y^3} - \frac{(\nabla y)^2}{y^2} = -\frac{1}{y^2}
\end{aligned}$$

using Lemma 3.3.7 in the last equality. Now, since $\delta y = 0$, we have that the scalar curvature is constant on the 2-surface, and positive, which means that its induced metric is locally the standard metric for S^2 with radius $|y|$. Now, since $\nabla y \neq 0$ on our open set, it is possible to choose a local coördinate system $\{x^0, y, x^2, x^3\}$ compatible with the foliation. Looking at (3.7) we see that t^a is non-vanishing inside $\mathcal{M}_{\underline{l}}$, and is in fact tangent to the 2-surface formed by $\{l, \underline{l}\}$, so we can take $t = t_x \partial_{x^0}$ for some function t_x. The fact that t^a is Killing gives that $\partial_{x^A} t_x = 0$ for $A = 2, 3$. Recall that we are working in $\mathcal{M}_{\underline{l}}$, and we assumed that $\underline{l}_a t^a = 1$, then we can write, by (3.16), $\underline{l} = \partial_y + s_x \partial_{x^0}$ for some function s_x. The commutator identity

$$[D, \delta] = -(\Gamma_{124} + \bar{\theta})\delta + \zeta D$$

shows that $\partial_{x^A} s_x = 0$ by considering the decomposition we have for $\underline{l}$ in terms of the coördinate vector fields. Then the Killing relation $[t, \underline{l}] = 0$, together with the above, implies that we can chose a coördinate system $\{u, y, x^2, x^3\}$ with $\partial_u = t$ and $\partial_y = \underline{l}$ that is compatible with the foliation. Lastly, we want to calculate $g_{AB} = g(\partial_{x^A}, \partial_{x^B})$ in this coördinate system. To do so, we use the fact that

$$-l^a = t^a + U\underline{l}^a$$

Then the second fundamental form can be written as

$$\begin{aligned}
\chi(X, Y) &= (\nabla_X Y)^\perp \\
&= -(\nabla_X Y)^a (\underline{l}_a l^b + l_a \underline{l}^b) \\
&= -(\nabla_X Y)^a (\underline{l}_a l^b - (t_a + U\underline{l}_a)\underline{l}^b)
\end{aligned}$$

Now, when X, Y are tangential fields, since U only depends on y (recall that $z = \mathfrak{A} = 0$), we have that $\nabla_X U = 0$. Furthermore, we use $g(Y, l) = g(Y, t) = 0$ to see

$$\chi(X, Y)^b = l^b Y^a X^c \nabla_c \underline{l}_a - \underline{l}^b Y^a X^c \nabla_c t_a - \underline{l}^b U Y^a X^c \nabla_c \underline{l}_a$$

So we have, using the fact that the second fundamental form is symmetric

$$\begin{aligned}2\chi(X,Y)^b &= Y^a X^c (l^b - U\underline{l}^b)\mathcal{L}_{\underline{l}} g_{ac} - Y^a X^c \underline{l}^b \mathcal{L}_t g_{ac} \\ &= -Y^a X^c \mathcal{L}_{\underline{l}} g_{ac} \nabla^b y\end{aligned}$$

Taking X and Y to be coördinate vector fields, we conclude that

$$\partial_y g_{AB} = \frac{2}{y} g_{AB}$$

so that $g_{AB} = y^2 g^0_{AB}$ where g^0_{AB} only depends on x^2, x^3. Imposing the condition that g_{AB} be the matrix for the standard metric on a sphere of radius $|y|$, we finally conclude that the line element can be written as

$$ds^2 = -(1 - \frac{2C_1\bar{C}_2 y - |C_1|^2}{y^2})du^2 + 2dudy + y^2 d\omega_{\mathbb{S}^2}$$

and thus the neighborhood can be embedded into Reissner-Nordström space-time of mass $C_1\bar{C}_2$ and charge $|C_1|$. □

Notice that a priori there is no guarantee that $C_1\bar{C}_2 y > 0$, this is compatible with the fact that we did not specify, for the local version of the theorem, the requirement for asymptotic flatness, and hence are in a case where the mass is not necessarily positive.

Next we consider the general case where t^a is not hypersurface orthogonal. In view of Proposition 3.3.9, we can assume that $\mathfrak{A} > 0$ and z not locally constant on any open set. Then it is clear that the set $\mathcal{M}_{\mathfrak{A}}$ is in fact dense in $\mathcal{M}$: for if there exists an open set on which $z = \mathfrak{A}$, then Proposition 3.3.9 implies that $\mathfrak{A} = 0$ identically on $\mathcal{M}$. Therefore, the set $(\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l) \cap \mathcal{M}_{\mathfrak{A}}$ is non-empty as long as $\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l$ is non-empty; this latter fact can be assured since by assumption (L) that t^a is timelike at some point $p \in \mathcal{M}$, whereas l^a and $\underline{l}^a$ are non-coïncidental null vectors, so in a neighborhood of p, we must have $l^a t_a \neq 0 \neq \underline{l}^a t_a$. It is on this set that we consider the next proposition.

Proposition 3.3.11. *Assuming $\mathfrak{A} > 0$. Let $p \in \mathcal{U} \subset \mathcal{M}_{\underline{l}} \cap \mathcal{M}_{\mathfrak{A}}$ such that t^a, n^a, b^a and $\underline{l}^a$ are well-defined on $\mathcal{U}$, with normalization $\underline{l}^a t_a = 1$. Then the four vector fields form a holonomic basis, and U can be isometrically embedded into a Kerr-Newman space-time.*

Before giving the proof, we first record the metric for the Kerr-Newman solution in Kerr coördinates

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr - q^2}{r^2 + a^2\cos^2\theta}\right) dV^2 + 2drdV + (r^2 + a^2\cos^2\theta)d\theta^2 \\ &+ \frac{\left[(r^2+a^2)^2 - (r^2 - 2Mr + a^2 + q^2)a^2\sin^2\theta\right]\sin^2\theta}{r^2 + a^2\cos^2\theta} d\phi^2 \\ &- 2a\sin^2\theta d\phi dr - \frac{2a(2Mr - q^2)}{r^2 + a^2\cos^2\theta}\sin^2\theta dV d\phi \end{aligned} \tag{3.18}$$

Notice that the metric is regular at $r = M \pm \sqrt{M^2 - a^2 - q^2}$ the event and Cauchy horizons.

Proof. We first note that in $\mathcal{M}_{\underline{l}}$, we have the normalization

$$n^a = (y^2 + z^2)(l^a + U\underline{l}^a) + (\mathfrak{A} + y^2)t^a$$

For the proof, it suffices to establish that the commutators between $n^a, b^a, \underline{l}^a, t^a$ vanish and that the vectors are linearly independent (for holonomy), and to calculate their relative inner products to verify that they define a coördinates equivalent to the Kerr coördinate above.

First we show that the commutators vanish. The cases $[t, \cdot]$ are trivial. Since we fixed $\underline{l}^a t_a = 1$, we have that

$$0 = t^b \nabla_b(\underline{l}_a t^a) = K_t t_b \underline{l}^b = K_t$$

so that $K_t = 0$ and thus $[t, \underline{l}] = [t, l] = 0$. Since y and z are geometric quantities defined from $\mathcal{H}_{ab}$, and U is a function only of y and z, they are symmetric under the action of t^a, therefore $[t, n] = 0$. Similarly, to evaluate $[t, b]$, it suffices to consider $[t, \nabla z]$. Using (3.5) we see that ∇z is defined by the volume form, the metric, and the vectors $t^a, \underline{l}^a, l^a$, all of which symmetric under t-action, and thus $[t, b] = 0$. The remaining cases require consideration of the connection coefficients. In view of the normalization condition imposed, $\nabla_a y = -l_a + U\underline{l}_a$, so (3.10a) implies $\underline{\theta}\bar{C}_1\bar{P} = 1$, $\theta C_1 P = -U$. Recall the null structure equation

$$-\delta\underline{\theta} = -\zeta\underline{\theta} + \underline{\eta}(\underline{\theta} - \bar{\underline{\theta}})$$

Using

$$0 = \delta(\underline{\theta}\bar{C}_1\bar{P}) = (\delta\underline{\theta})\bar{C}_1\bar{P} + \underline{\theta}\underline{\eta}\bar{C}_1\bar{P}$$

we have

$$\bar{C}_1\bar{P}(\underline{\theta}\underline{\eta} + \zeta\underline{\theta} - \underline{\eta}\underline{\theta} + \underline{\eta}\bar{\underline{\theta}}) = 0$$

Applications of (3.11) allow us to replace $+\underline{\eta}\bar{\underline{\theta}}$ by $-\eta\underline{\theta}$ in the brackets, and so, since $\underline{\theta}\bar{C}_1\bar{P} = \underline{D}C_1P \neq 0$, we must have $\zeta = \eta$, which considerably simplifies calculations. Next we write

$$b^a = -i\frac{y^2 + z^2}{\mathfrak{A} - z^2}(\eta C_1 P\bar{m}^a - \bar{\eta}\bar{C}_1\bar{P}m^a) = i\frac{1}{\mathfrak{A} - z^2}\left(\underline{\eta}C_1\bar{C}_1^2 P\bar{P}^2\bar{m}^a - c.c.\right)$$

by expanding $\nabla^a z$ in tetrad coefficients, and where *c.c.* denotes complex conjugate. Then, since $\underline{D}z = 0$,

$$-i(\mathfrak{A} - z^2)[\underline{l}, b] = \underline{D}(\underline{\eta}C_1\bar{C}_1 P\bar{P}^2)\bar{m}^a - c.c. + \underline{\eta}C_1\bar{C}_1^2 P\bar{P}^2[\underline{D}, \bar{\delta}] - c.c.$$

We consider the commutator relation, simplified appropriately in view of computations above and in the proof of Lemma 3.3.7,

$$[\underline{D}, \bar{\delta}] = -(\Gamma_{213} + \underline{\theta})\bar{\delta} = (\Gamma_{123} - \frac{1}{\bar{C}_1 \bar{P}})\bar{\delta}$$

together with the structure equation

$$(\underline{D} + \Gamma_{123})\underline{\eta} = \underline{\theta}(\eta - \underline{\eta})$$

and the relations in (3.11) and (3.9), we get

$$\begin{aligned}
\underline{D}(\underline{\eta} C_1 \bar{C}_1^2 P \bar{P}^2)\bar{m}^a &+ \underline{\eta} C_1 \bar{C}_1^2 P \bar{P}^2 [\underline{D}, \bar{\delta}] \\
&= (\underline{D} + \Gamma_{123})\underline{\eta} C_1 \bar{C}_1^2 P \bar{P}^2 \bar{m}^a - \underline{\eta}|C_1 P|^2 \bar{m}^a + \underline{\eta}\underline{D}(C_1 \bar{C}_1^2 P \bar{P}^2)\bar{m}^a \\
&= \underline{\theta}(\eta - \underline{\eta}) C_1 \bar{C}_1^2 P \bar{P}^2 \bar{m}^a - \underline{\eta}|C_1 P|^2 \bar{m}^a \\
&\qquad + \underline{\eta}(\underline{\theta} \bar{C}_1^3 \bar{P}^3 + 2\underline{\theta} C_1 \bar{C}_1^2 P \bar{P}^2)\bar{m}^a \\
&= 0
\end{aligned}$$

Hence $[\underline{l}, b] = 0$. In a similar fashion, we write

$$n^a = |C_1 P|^2 l^a + \frac{1}{2}(\mathfrak{A} + y^2 + |C_1|^2 - 2C_1 \bar{C}_2 y)\underline{l}^a + (\mathfrak{A} + y^2)t^a$$

From the fact that $b^a \nabla_a y = 0$ and from the known commutator relations, we have

$$[n, b] = [C_1 \bar{C}_1 P \bar{P} l, b] + \frac{1}{2}(\mathfrak{A} + y^2 + |C_1|^2 - 2C_1 \bar{C}_2 y)[\underline{l}, b] + (\mathfrak{A} + y^2)[t, b]$$

of which the second and third terms are already known to vanish. We evaluate $[C_1 \bar{C}_1 P \bar{P} l, b]$ in the same way we evaluated $[\underline{l}, b]$, and a calculation shows that it also vanishes. To evaluate $[\underline{l}, n]$, we need to calculate $[\underline{l}, l]$. To do so we write

$$t^a = -U\underline{l}^a - l^a - \bar{\eta} C_1 P m^a - \eta \bar{C}_1 \bar{P} \bar{m}^a$$

Since $[\underline{l}, t] = 0$, we infer

$$[\underline{l}, l] = -[\underline{l}, U\underline{l} + \bar{\eta}C_1 P m + \eta \bar{C}_1 \bar{P}\bar{m}]$$
$$= -\underline{D}U\underline{l} - [\underline{l}, \frac{1}{|C_1 P|^2}\bar{\eta}C_1^2\bar{C}_1 P^2 \bar{P} m] - c.c.$$

Notice in the proof above for $[\underline{l}, b] = 0$ we have demonstrated $[\underline{l}, \bar{\eta}C_1^2\bar{C}_1 P^2\bar{P}m] = 0$, so

$$[\underline{l}, l] = -\underline{D}U\underline{l} + \frac{\underline{D}(|C_1 P|^2)}{|C_1 P|^2}(\bar{\eta}C_1 P m + \eta\bar{C}_1\bar{P}\bar{m})$$

Direct computation yields

$$\underline{D}U = \frac{y - C_1\bar{C}_2}{y^2 + z^2} - \frac{2yU}{y^2 + z^2}$$

and

$$\underline{D}(C_1\bar{C}_1 P\bar{P}) = 2y$$

(recall that we set $\underline{D}y = 1$) so we conclude that

$$[\underline{l}, l] = -\frac{y - C_1\bar{C}_2}{y^2 + z^2}\underline{l} - \frac{2y}{y^2 + z^2}(l + t)$$

So, using the decomposition for n^a given above

$$[\underline{l}, n] = [\underline{l}, (y^2 + z^2)l + (y^2 + z^2)U\underline{l} + (\mathfrak{A} + y^2)t]$$
$$= 2yl + (y - 2C_1\bar{C}_2)\underline{l} + 2yt + (y^2 + z^2)[\underline{l}, l]$$
$$= 0$$

Having checked the commutators, we now calculate the scalar products between various components. A direct computation from the definition yields

$$b^2 = \frac{y^2+z^2}{\mathfrak{A}-z^2} \qquad b\cdot n = 0 \qquad b\cdot \underline{l} = 0$$

$$\underline{l}\cdot n = \mathfrak{A}-z^2 \qquad \underline{l}^2 = 0 \qquad \underline{l}\cdot t = 1$$

$$t\cdot n = \frac{(|C_1|^2 - 2C_1\bar{C}_2 y)(z^2-\mathfrak{A})}{y^2+z^2} \qquad t^2 = -1 - \frac{|C_1|^2 - 2C_1\bar{C}_2 y}{y^2+z^2} \qquad t\cdot b = 0$$

and

$$n^2 = (\mathfrak{A}-z^2)\left[\mathfrak{A} + y^2 - \frac{\mathfrak{A}-z^2}{y^2+z^2}\left(|C_1|^2 - 2C_1\bar{C}_2 y\right)\right]$$

A simple computation shows that the determinant of the matrix of inner products yields

$$|\det| = (y^2+z^2)^2 \neq 0$$

and therefore the vector fields are linearly independent. Thus we have shown that they form a holonomic basis.

To construct the local isometry to Kerr-Newman space-time, we define coördinates attached to the holonomic vector fields $t, \underline{l}, b, n$ with the following rescalings. First, since $\mathfrak{A} > 0$, we can define $a > 0$ such that $\mathfrak{A} = a^2$. Then we can define the coördinates r, θ, V, ϕ by

$$t = \partial_V$$

$$\underline{l} = \partial_r \qquad y = r$$

$$b = \frac{1}{a\sin\theta}\partial_\theta \qquad z = a\cos\theta$$

$$n = -a\partial_\phi$$

Notice that we can define θ from z in a way that makes sense since $z^2 \leq \mathfrak{A}$. Applying the change of coördinates to the inner products above we see that in r, θ, V, ϕ the metric is identical to the one for the Kerr coördinate of Kerr-Newman space-time.

Furthermore, we see that n, or ∂_ϕ, defines the corresponding axial Killing vector field. $\square$

To finish this section, we need to show that the results we obtained in Propositions 3.3.10 and 3.3.11 can be extended to the manifold $\mathcal{M}$, rather than restricted to $(\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l)$ in the former and $(\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l) \cap \mathcal{M}_{\mathfrak{A}}$ in the latter. We shall need the following lemma (Lemma 6 in [22]; the lemma and its proof can be carried over to our case essentially without change, we reproduce them here for completeness)

Lemma 3.3.12. *The vector field n^a is a Killing vector field on the entirety of $\mathcal{M}$. The set $\mathcal{M} \setminus \mathcal{M}_{\mathfrak{A}} = \{n^a = 0\}$. Furthermore,*

- *If $\mathfrak{A} = 0$, then $\mathcal{M} \setminus (\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l) = \{t^a = 0\}$*
- *If $0 < \mathfrak{A} \leq (C_1\bar{C}_2)^2 - |C_1|^2$, then $\mathcal{M} \setminus (\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l) = \{$ either $n^a - y_+ t^a = 0$ or $n^a - y_- t^a = 0\}$ where*

$$y_\pm = 2(C_1\bar{C}_2)^2 - |C_1|^2 \pm 2C_1\bar{C}_2\sqrt{(C_1\bar{C}_2)^2 - |C_1|^2 - \mathfrak{A}}$$

- *If $\mathfrak{A} > (C_1\bar{C}_2)^2 - |C_1|^2$, then $\mathcal{M} \setminus (\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l) = \emptyset$*

Proof. First consider the case $\mathfrak{A} = 0$. By Proposition 3.3.9, we have $z = 0$. So the definition (3.14) and (3.7) show that n^a vanishes identically. Furthermore, since $\mathcal{M}_{\mathfrak{A}} = \emptyset$ in this case, we have that n^a is a (trivial) Killing vector field on $\mathcal{M}$ vanishing on $\mathcal{M} \setminus \mathcal{M}_{\mathfrak{A}}$. It is also clear from (3.7) that $t^a = 0 \iff t_a l^a = t_a \underline{l}^a = 0$ in this case, proving the first bullet point.

Now let $\mathfrak{A} > 0$. Then Proposition 3.3.11 shows that n^a is Killing on $(\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l) \cap \mathcal{M}_{\mathfrak{A}}$, and does not coïncide with t^a. Since $\mathcal{M}_{\mathfrak{A}}$ is dense in $\mathcal{M}$ (see paragraph immediately before Proposition 3.3.11), we have that n^a is Killing on $\overline{\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l}$ (the overline denotes set closure). We wish to show that $\overline{\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l} = \mathcal{M}$. Suppose not, then the open set $\mathcal{U} = \mathcal{M} \setminus \overline{\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l}$ is non-empty. In $\mathcal{U}$, $t_a l^a = t_a \underline{l}^a = 0$, so by (3.5),

$\nabla^a y = 0$ in $\mathcal{U}$. Taking the real part of the third identity in Proposition 3.3.4, we must have $y = C_1\bar{C}_2$ in $\mathcal{U}$, which by Lemma 3.3.7 implies $\mathfrak{A} = (C_1\bar{C}_2)^2 - |C_1|^2$. Consider the vectorfield defined on all of $\mathcal{M}$ given by $n^a - (\mathfrak{A}+y^2)t^a = n^a - [2(C_1\bar{C}_2)^2 - |C_1|^2]t^a$. As it is a constant coefficient linear combination of non-vanishing independent Killing vector fields on $\overline{\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l}$, it is also a non-vanishing Killing vector field. However, on $\mathcal{U}$, the vector field vanishes by construction. So we have Killing vector field on $\mathcal{M}$ that is not identically 0, yet vanishes on an non-empty open set, which is impossible (see Appendix C.3 in [45]). Therefore n^a is a Killing vector field everywhere on $\mathcal{M}$. Now, outside of $\mathcal{M}_{\mathfrak{A}}$, we have that $z^2 = \mathfrak{A}$ reaches a local maximum, so $\nabla_a z$ must vanish. Therefore from (3.14) and (3.7) we conclude that n^a vanishes outside $\mathcal{M}_{\mathfrak{A}}$ also, proving the second statement in the lemma.

For the second a third bullet points, consider the function $U = \frac{1}{2}(\nabla y)^2$. By definition it vanishes outside $\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l$. Using Lemma 3.3.7 we see that

$$\mathfrak{A} + y^2 + |C_1|^2 - 2C_1\bar{C}_2 y = 0$$

outside $\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l$. The two bullet points are clear in view of the quadratic formula and (3.14). □

Now we can complete the main theorem in the same way as [22].

Proof of the Main Theorem. In view of Propositions 3.3.10 and 3.3.11, we only need to show that the isometry thus defined extends to $\mathcal{M} \setminus (\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l)$ in the case of Reissner-Nordström and $\mathcal{M} \setminus [(\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l) \cap \mathcal{M}_{\mathfrak{A}}]$ in the case of Kerr-Newman. Lemma 3.3.12 shows that those points we are interested in are fixed points of Killing vector fields, and hence are either isolated points or smooth, two-dimensional, totally geodesic surfaces. Their complement, therefore, are connected and dense, with local isometry into the Kerr-Newman family. Therefore a sufficiently small neighborhood of one of these fixed-points will have a dense and connected subset isometric to a patch of Kerr-Newman, whence we can extend to those fixed-points by continuity. □

3.4 Proof of the main global result

To show Corollary 3.2.2, it suffices to demonstrate that the global assumption (G) leads to the local assumption (L).

By asymptotic flatness and the imposed decay rate (the assumption that the mass and charge at infinity are non-zero), we can assume that there is a simply connected region $\mathcal{M}_{\mathcal{H}}$ near spatial infinity such that $\mathcal{H}^2 \neq 0$. It thus suffices to show that $\mathcal{M}_{\mathcal{H}} = \mathcal{M}$. Suppose not, then the former is a proper subset of the latter. Let $p_0 \in \mathcal{M}$ be a point on $\partial\mathcal{M}_{\mathcal{H}}$. We see that Theorem 3.2.1 applies to $\mathcal{M}_{\mathcal{H}}$, with C_1 taken to be $q_E + iq_B$ and $C_3 = M/(q_E - iq_B)$. In particular, the first equation in Proposition 3.3.4 shows that, by continuity, $t^2 = -1$ at p_0. Let δ be a small neighborhood of p_0 such that t^a is everywhere time-like in δ with $t^2 < -\frac{1}{4}$, then the metric g induces a uniform Riemannian metric on the bundle of orthogonal subspaces to t^a, i.e. $\cup_{p\in\delta}\{v \in T_p\mathcal{M} | g(v,t) = 0\}$. Now, consider a curve $\gamma : (s_0, 1] \to \delta$ such that $\gamma(s) \in \mathcal{M}_{\mathcal{H}}$ for $s < 1$, $\gamma(1) = p_0$, and $\frac{d}{ds}\gamma(s)$ has norm 1 and is orthogonal to t. Consider the function $(q_E + iq_B)P \circ \gamma$. By assumption, $|(q_E + iq_B)P \circ \gamma| \nearrow \infty$ as $s \nearrow 1$. Since Lemma 3.3.7 guarantees that z is bounded in $\mathcal{M}_{\mathcal{H}}$, and hence by continuity, at p_0, we must have that y blows up as we approach p_0 along γ. However,

$$|\frac{d}{ds}(y \circ \gamma)| = |\nabla_{\frac{d}{ds}\gamma} y| \leq C\sqrt{|\nabla_a y \nabla^a y|} < C' < \infty$$

where the constant C comes from the uniform control on g acting as a Riemannian metric on the orthogonal subspace to t^a (note that $t^a\nabla_a y = 0$ since y is a quantity derivable from quantities that are invariant under the t-action), and C' arises because by Lemma 3.3.7, $\nabla_a y \nabla^a y$ is bounded for all $|y| > 2M$, which we can guarantee for s sufficiently close to 1. So we have a contradiction: $y \circ \gamma$ blows up in finite time while its derivative stays bounded. Therefore $\mathcal{M}_{\mathcal{H}} = \mathcal{M}$.

3.5 Reduction to the Kerr case

The results stated in Corollary 3.2.2 explicitly makes the assumption that the asymptotic charge of the space-time does not vanish. With a small modification to the statement, combined with the result of Mars [22], we can restate the result to also include the case of vanishing charge.

Theorem 3.5.1. *Let $(\mathcal{M}, g_{ab}, H_{ab})$ solve the Einstein-Maxwell system. Assuming (A1), (A2), and that $(\mathcal{M}, g_{ab})$ contains a stationary asymptotically flat end $\mathcal{M}^\infty$ where t^a tends to a time translation, with non-vanishing Komar mass M. Let Ξ' be the complex-valued scalar function defined by*

$$\nabla_a \Xi' = (q_E - iq_B)\mathcal{H}_{ba} t^b$$

where q_E and q_B are the asymptotic electric and magnetic charges respectively, and Ξ' normalized such that it approaches 0 at spatial infinity. Assume there exists a complex-valued function P' defined whenever $\mathcal{F}^2 \neq 0$ such that

$$(P')^{-4} = -\frac{\mathcal{F}^2}{(4\bar{\Xi}' - 2M)^2} \ .$$

If the double-alignment conditions

$$(q_E + iq_B)\mathcal{F}_{ab} = (4\bar{\Xi}' - 2M)\mathcal{H}_{ab}$$
$$\mathcal{C}_{abcd} = \frac{3P'}{4\bar{\Xi}' - 2M}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{abcd}$$

are satisfied whenever the right-hand-sides are well-defined, then we can conclude that

1. *either $\mathcal{H}^2$ is non-vanishing globally, or that $\mathcal{H}_{ab} = 0$ everywhere,*

2. $\mathfrak{A} = P'\bar{P}'(\Im\nabla P')^2 + (\Im P')^2$ *is a non-negative real constant on the manifold,*

3. and $(\mathcal{M}, g_{ab})$ *is everywhere locally isometric to a Kerr-Newman space-time of total charge* $q = \sqrt{q_E^2 + q_B^2}$*, angular momentum* $\sqrt{\mathfrak{A}}M$*, and mass* M*.*

Proof. In the cases where $q > 0$, it is clear to see that with the substitution $\Xi' = (q_E - iq_B)\Xi$ and $P' = (q_E + iq_B)P$, the definition of P' and the double-alignment conditions become equivalent to the assumptions made in the statement of Corollary 3.2.2. And hence the space-time is locally isometric to a Kerr-Newman space-time of non-vanishing charge.

The renormalization imposed in the statement of this theorem allows us to incorporate the result of Mars. Note that by the stated conditions, if $q = 0$, then the scalar function Ξ' vanishes identically on the space-time. The first alignment condition implies then that

$$0 \cdot \mathcal{F}_{ab} = -2M\mathcal{H}_{ab} \ ,$$

guaranteeing that $\mathcal{H}_{ab}$ vanishes identically on the space-time. Then P' can be identified with the scalar P appearing in [22], and that the second alignment condition is precisely the alignment condition given in [22] or in [14] (up to factors of 2 which arises from differing definition of the anti-self-dual projection and from the definition of the Ernst two-form). Therefore we are allowed to apply Mars' theorem (or one can directly modify the proof of Theorem 3.2.1 to account for the renormalization) and conclude that the space-time is locally isometric to Kerr. □

In view of the above reduction, we collect the definition of the renormalized versions of $\mathcal{B}_{ab}$, $\mathcal{Q}_{abcd}$, and P below. These three quantities will be useful in the next chapter.

$$\mathcal{B}'_{ab} := (q_E + iq_B)\mathcal{F}_{ab} - (4\bar{\Xi}' - 2M)\mathcal{H}_{ab} \tag{3.19a}$$

$$\mathcal{Q}'_{abcd} := \mathcal{C}_{abcd} - \frac{3P'}{4\bar{\Xi}' - 2M}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{abcd} \tag{3.19b}$$

$$(P')^{-4} := -\frac{\mathcal{F}^2}{(4\bar{\Xi}' - 2M)^2}\ . \tag{3.19c}$$

Chapter 4

Uniqueness of the Kerr-Newman solutions

The strategy for obtaining uniqueness is to rely on Carleman-estimate techniques as in [14, 15]. The tensor quantities $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ will be seen to verify essentially decoupled wave equations, and by making some technical assumptions on the bifurcate sphere, $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ can be seen to vanish on the bifurcate event horizon of the space-time. By applying the Carleman estimate, the two tensors must vanish throughout the manifold. As seen in Theorem 3.5.1 from the previous chapter, the vanishing of the tensors $\mathcal{Q}'_{abcd}$ and $\mathcal{B}'_{ab}$ allows the construction of local isometries from the given space-time into a Kerr-Newman space-time.

4.1 The wave equations for $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$

For the application of Carleman estimates, it is essential that the tensor quantities $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ verify sourceless wave equations, which can be written schematically as

$$\Box_g \mathcal{S} = \upsilon_1 \mathcal{A}_1 \circledast \mathcal{S} + \upsilon_2 \mathcal{A}_2 \circledast \nabla \mathcal{S} \tag{4.1}$$

where, if $\mathcal{S}$ is a (p,q) tensor field, $\mathcal{A}_1$ is type $(p+q, q+p)$, $\mathcal{A}_2$ is type $(p+q+1, q+p)$, and $\circledast$ denotes full contraction of all the indices of $\mathcal{S}$ against those of $\mathcal{A}_*$. We will let $\mathcal{A}_*$ be a smooth tensor field constructed of linear combinations of tensor products and contractions of (assumed) smooth geometric quantities, such as $\mathcal{C}_{abcd}, \mathcal{F}_{ab}, \mathcal{H}_{ab}, \Xi'$ and their covariant derivatives. υ_* are scalar coefficients that absorbs the not-a-priori-smooth terms, such as P' and $\frac{1}{4\Xi'-2M}$. The exact form of $\mathcal{A}_*$ are not too important; the form of υ_* are more so in the application of the Carleman inequality. The crucial information, however, is that (4.1) is *order reducing* (that a second order operator acting on $\mathcal{S}$ only gives rise to terms of zeroth or first order terms in $\mathcal{S}$) and at least linear in $\mathcal{S}$ (that it doesn't contain terms that does not depend on $\mathcal{S}$ or $\nabla\mathcal{S}$ tensorially; note that the dependence can be quadratic or even higher power: it suffices that a linear factor can be taken out).

Proposition 4.1.1. *$\mathcal{B}'_{ab}$ satisfies a wave equation of the form (4.1).*

Proof. By the computation (2.19), $\mathcal{B}_{ab}$, and hence $\mathcal{B}'_{ab}$, is Maxwell. So using the same calculation as (2.17),

$$\Box_g \mathcal{B}'_{ab} = -\mathcal{C}_{abcd}\mathcal{B}'^{cd} \tag{4.2}$$

as claimed. □

Notice that the wave equation for $\mathcal{B}'_{ab}$ is decoupled completely from $\mathcal{Q}'_{abcd}$. This fact will become useful later. For now, just observe that the above proposition implies that

$$\Box_g \nabla^{(k)}\mathcal{B}' = \sum_{l=0}^{k} \mathcal{A}_l \circledast \nabla^{(l)}\mathcal{B}' \ . \tag{4.3}$$

For the field $\mathcal{Q}'_{abcd}$, the wave equation is coupled to higher derivatives of $\mathcal{B}'_{ab}$. The calculation below is rather *ad hoc*, and depends on some rather miraculous algebraic cancellations. Currently, there is no justification, formally or heuristically, on why such a wave equation should be possible. Indeed, a more detailed understanding of the mechanism through which this wave equation is obtained may greatly contribute to our understanding of stationary black holes.

Proposition 4.1.2. *Where it is defined, $\mathcal{Q}'_{abcd}$ satisfies a wave equation of the form*

$$\Box_g \mathcal{Q}' = \upsilon_1 \mathcal{A}_1 \circledast \mathcal{Q}' + \upsilon_2 \mathcal{A}_2 \circledast \nabla \mathcal{Q}' + \upsilon_3 \mathcal{A}_3 \circledast \mathcal{B}' + \upsilon_4 \mathcal{A}_4 \circledast \nabla \mathcal{B}' + \upsilon_5 \mathcal{A}_5 \circledast \nabla^2 \mathcal{B}'$$

First it is claimed that it suffices to demonstrate a divergence type equation.

Lemma 4.1.3. *Let $\mathcal{S}_{abcd}$ be an anti-self-dual Weyl field which satisfies*

$$\nabla^a \mathcal{S}_{abcd} = \mathcal{J}_{bcd} \ , \tag{4.4}$$

where $\mathcal{J}_{bcd}$ is some source term, then $\mathcal{S}_{abcd}$ satisfies an inhomogeneous wave equation.

Proof. Since $\mathcal{S}_{abcd}$ is anti-self-dual,

$$\nabla^a \mathcal{S}_{abcd} = \frac{i}{2} \nabla^a \epsilon_{abef} \mathcal{S}^{ef}{}_{cd} \ ,$$

which implies

$$\nabla_{[e} \mathcal{S}_{ab]cd} = -\frac{i}{3} \epsilon_{eab}{}^{k} \mathcal{J}_{kcd} = \mathcal{J}'_{eabcd} \ .$$

Take the divergence

$$\Box_g \mathcal{S}_{abcd} + \nabla^e \nabla_a \mathcal{S}_{becd} + \nabla^e \nabla_b \mathcal{S}_{eacd} = \nabla^e \mathcal{J}'_{eabcd}$$

and commuting derivatives

$$\Box_g \mathcal{S}_{abcd} = \nabla^e \mathcal{J}'_{eabcd} + [\nabla_a, \nabla^e] \mathcal{S}_{becd} + [\nabla_b, \nabla^e] \mathcal{S}_{eacd} + \nabla_a \mathcal{J}_{bcd} - \nabla_b \mathcal{J}_{acd}$$

as claimed. □

We also have a small computational lemma which will also be useful independent of the proof of the proposition.

Lemma 4.1.4. *The scalar P', whenever it is well-defined, satisfies*

$$-\nabla_c P' = \frac{P'}{\mathcal{F}^2} \mathcal{C}_{dcab} t^d \mathcal{F}^{ab} \ . \tag{4.5}$$

Proof. Recall $\mathcal{F}^2 P'^4 = -(4\bar{\Xi}' - 2M)^2$,

$$\mathcal{F}^{ab} \nabla_c \mathcal{F}_{ab} P'^4 + 2\mathcal{F}^2 P'^3 \nabla_c P' = -4(4\bar{\Xi}' - 2M)(q_E + iq_B)\bar{\mathcal{H}}_{dc} t^d \ ,$$

using (2.18),

$$-\nabla_c P' = \frac{P'}{\mathcal{F}^2}(\mathcal{C}_{dcab} + \mathcal{E}_{dcab})\mathcal{F}^{ab} t^d + \frac{2(4\bar{\Xi}' - 2M)(q_E + iq_B)}{P'^3 \mathcal{F}^2} \bar{\mathcal{H}}_{dc} t^d \ .$$

Observe that

$$\begin{aligned}
\mathcal{E}^{waxy}\mathcal{F}_{xy} &= \frac{1}{2}(g \owedge T)^{waxy}(\mathcal{P}_{-}\mathcal{F})_{xy} = T^w_x \mathcal{F}^{xa} + T^a_y \mathcal{F}^{wy} \\
&= 4(\bar{\mathcal{H}}^{wz}\mathcal{H}_{xz}\mathcal{F}^{xa} + \mathcal{H}^{az}\bar{\mathcal{H}}_{yz}\mathcal{F}^{wy}) \\
&= \frac{4\bar{\mathcal{H}}^{wz}((q_E + iq_B)\mathcal{F}_{xz} - \mathcal{B}'_{xz})\mathcal{F}^{xa}}{4\bar{\Xi}' - 2M} + \frac{4\mathcal{H}^{az}\bar{\mathcal{H}}_{yz}(\mathcal{B}'^{wy} + (4\bar{\Xi}' - 2M)\mathcal{H}^{wy})}{q_E + iq_B} \\
&= \frac{4\bar{\mathcal{H}}^{wz}\mathcal{F}^{xa}}{4\bar{\Xi}' - 2M}\mathcal{B}'_{zx} + \frac{4\mathcal{H}^{az}\bar{\mathcal{H}}_{yz}}{q_E + iq_B}\mathcal{B}'^{wy} + \frac{\bar{\mathcal{H}}^{wa}(q_E + iq_B)\mathcal{F}^2}{4\bar{\Xi}' - 2M} + \frac{\mathcal{H}^2\bar{\mathcal{H}}^{wa}(4\bar{\Xi}' - 2M)}{q_E + iq_B}
\end{aligned}$$

Since

$$(4\bar{\Xi}' - 2M)^2\mathcal{H}^2 = (q_E + iq_B)^2\mathcal{F}^2 + \mathcal{B}'^2 - 2(q_E + iq_B)\mathcal{F} \cdot \mathcal{B}$$

we write

$$\begin{aligned}
\mathcal{E}^{waxy}\mathcal{F}_{xy} = -\frac{4\bar{\mathcal{H}}^{wz}\mathcal{F}^{xa}}{4\bar{\Xi}' - 2M}\mathcal{B}'_{xz} + \frac{4\bar{\mathcal{H}}^{az}\mathcal{F}_{yz}}{4\bar{\Xi}' - 2M}\mathcal{B}'^{wy} + \frac{2\bar{\mathcal{H}}^{wa}\mathcal{F}^{xy}}{4\bar{\Xi}' - 2M}\mathcal{B}'_{xy} \\
- \frac{2(q_E + iq_B)(4\bar{\Xi}' - 2M)}{P'^4}\bar{\mathcal{H}}^{wa}
\end{aligned}$$

where in the second term on the right hand side we replaced

$$\mathcal{H}^{az}\bar{\mathcal{H}}_{yz} = \bar{\mathcal{H}}^{az}\frac{(q_E + iq_B)\mathcal{F}_{yz} - \mathcal{B}'_{yz}}{4\bar{\Xi}' - 2M}$$

and liberally used multiplication properties of anti-self-dual forms. Noting that by construction the expression should be antisymmetric in the w, a indices, one sees that by (2.4c), the first three terms on the right hand side sums to zero. So we have that

$$\mathcal{E}^{waxy}\mathcal{F}_{xy} = -\frac{2(q_E + iq_B)(4\bar{\Xi}' - 2M)}{P'^4}\bar{\mathcal{H}}^{wa} \ , \tag{4.6}$$

and

$$-\nabla_c P' = \frac{P'}{\mathcal{F}^2}\mathcal{C}_{dcab}t^d\mathcal{F}^{ab}$$

as desired. □

As seen briefly in the proof above, some basic strategies involved in the computation include (1) the ability to exchange $\mathcal{F}_{ab}$ and $\mathcal{H}_{ab}$ (up to scalar terms) by sacrificing $\mathcal{B}'_{ab}$ and (2) applications of multiplication properties of anti-self-dual forms (as discussed in Section 2.1.1). The best example of the strategies is the derivation of (4.6) above. The most difficult term in the computation of $\nabla^a\mathcal{Q}'_{abcd}$ turns out to be the divergence of the Weyl curvature, which involves derivatives of the stress-energy tensor. This term gives the only contribution (in the final expression) of terms involving $\nabla\mathcal{B}'$.

Proof. (Proposition 4.1.2) By Lemma 4.1.3, it suffices to show that under the stated conditions, $\mathcal{Q}'_{abcd}$ satisfies a divergence equation with source term depending linearly on $\mathcal{Q}'_{abcd}, \mathcal{B}'_{ab}$, and $\nabla_c\mathcal{B}'_{ab}$.

The goal is to calculate the divergence

$$\begin{aligned}\nabla^a \mathcal{Q}'_{abcd} &= \nabla^a \mathcal{C}_{abcd} - \frac{3\nabla^a P'}{4\bar{\Xi}' - 2M}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{abcd} + \frac{12P'(q_E + iqB)\bar{\mathcal{H}}^{xa}t_x}{(4\bar{\Xi}' - 2M)^2}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{abcd} \\ &\quad - \frac{3P'}{4\bar{\Xi}' - 2M}\left(2\mathcal{E}^{xa}{}_{ab}t_x\mathcal{F}_{cd} + \mathcal{F}_{ab}\nabla^a\mathcal{F}_{cd} - \frac{2}{3}\mathcal{I}_{abcd}\mathcal{F}^{ef}\nabla^a\mathcal{F}_{ef}\right) \\ &= T_1 + T_2 + T_3 - \frac{3P'}{4\bar{\Xi}' - 2M}(T_4 + T_5 + T_6)\end{aligned}$$

term by term. Immediately, by (4.5)

$$T_2 = \frac{3P'}{\mathcal{F}^2(4\bar{\Xi}' - 2M)}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{abcd}(\mathcal{Q}'^{waxy}t_w\mathcal{F}_{xy} + \frac{2P'\mathcal{F}^2}{4\bar{\Xi}' - 2M}\mathcal{F}^{wa}t_w) \ . \tag{4.7}$$

To consider T_1, recall from (2.15)

$$\nabla^a \mathcal{C}_{abcd} = \mathcal{I}_{cdgh}(\nabla^g T^h_b) \ ,$$

expanding from the right-hand side

$$\nabla^g T^h_b = 4\nabla^g(\mathcal{H}_{bk}\bar{\mathcal{H}}^{hk})$$

and since we act on it by the projection operator $\mathcal{P}_-$, it suffices to consider the anti-symmetric, anti-self-dual part of this expression. By Maxwell's equations

$$\begin{aligned}\frac{1}{2}\nabla^{[g}T^{h]}_b &= 2\nabla^{[g}\bar{\mathcal{H}}^{h]k}\mathcal{H}_{bk} - 2\bar{\mathcal{H}}^{k[h}\nabla^{g]}\mathcal{H}_{bk} \\ &= -\nabla^k\bar{\mathcal{H}}^{gh}\mathcal{H}_{bk} + \bar{\mathcal{H}}^{hk}\nabla^g\mathcal{H}_{bk} - \bar{\mathcal{H}}^{gk}\nabla^h\mathcal{H}_{bk} \\ &= \not{L}^{gh}_b + L^{gh}_b - L^{hg}_b \ .\end{aligned}$$

In the following computation, $\not{L}^{gh}_b$ will be continuously redefined to include all symmetric and self-dual parts of the expression, while L^{gh}_b will contain those terms of interest. It is clear that the first term on the second line in the equation above is self-dual, hence will be grouped into $\not{L}^{gh}$. It suffices to consider $L_b^{gh} = \bar{\mathcal{H}}^{hk}\nabla^g\mathcal{H}_{bk}$. Now,

$$\nabla^g\mathcal{H}_{bk} = \frac{-4(q_E + iq_B)\bar{\mathcal{H}}^{xg}t_x\mathcal{H}_{bk} + 2(q_E + iq_B)(\mathcal{C}^{xg}{}_{bk} + \mathcal{E}^{xg}{}_{bk})t_x - \nabla^g\mathcal{B}'_{bk}}{4\bar{\Xi}' - 2M} \tag{4.8}$$

So

$$L_b^{gh} = \frac{2(q_E + iq_B)(\mathcal{C}^{xg}{}_{bk} + \mathcal{E}^{xg}{}_{bk})t_x\bar{\mathcal{H}}^{hk} - (q_E + iq_B)T_b^h\bar{\mathcal{H}}^{xg}t_x - \bar{\mathcal{H}}^{hk}\nabla^g\mathcal{B}'_{bk}}{4\bar{\Xi}' - 2M} \ .$$

Consider the following simple identities:

$$T^{ad}\mathcal{H}_{ac} = 4\mathcal{H}^{ax}\bar{\mathcal{H}}_{dx}\mathcal{H}_{ac} = \mathcal{H}^2\bar{\mathcal{H}}_{dc} \tag{4.9}$$

$$i\bar{\mathcal{H}}^{hk}\epsilon_{wyzk} = \frac{1}{2}\epsilon^{hkml}\bar{\mathcal{H}}_{ml}\epsilon_{wyzk} = -3g^h_{[w}\bar{\mathcal{H}}_{yz]} \tag{4.10}$$

the second one implies that

$$4\mathcal{I}_{wyzk}\bar{\mathcal{H}}^{hk} = g_{wz}\bar{\mathcal{H}}^h{}_y - g_{yz}\bar{\mathcal{H}}^h{}_w - g^h_w\bar{\mathcal{H}}_{yz} + g^h_y\bar{\mathcal{H}}_{wz} + g^h_z\bar{\mathcal{H}}_{yw} = 4\mathcal{I}_{wy}{}^h{}_k\bar{\mathcal{H}}_z{}^k \ .$$

Therefore, with congruence $\cong$ up to terms that can be thrown into $\not{L}_b^{gh}$, we get

$$2\bar{\mathcal{H}}^{hk}\mathcal{E}^{dg}{}_{bk} \cong \frac{1}{2}[g^d_b\bar{\mathcal{H}}^2\mathcal{H}^{hg} - g^g_b\bar{\mathcal{H}}^2\mathcal{H}^{hd} - \bar{\mathcal{H}}^{hd}T^g_b + T^{dh}\bar{\mathcal{H}}_b{}^g + 2g^h_b\bar{\mathcal{H}}^2\mathcal{H}^{gd} + g^{dh}\bar{\mathcal{H}}^2\mathcal{H}_b{}^g] \ .$$

Now, to make use of the anti-symmetry in the g, h indices, note that

$$\mathcal{X}_{a[b}\mathcal{X}_{cd]} = \frac{1}{6}\epsilon_{bcde}\epsilon^{efgh}\mathcal{X}_{af}\mathcal{X}_{gh} = \frac{i}{12}\epsilon_{abcd}\mathcal{X}^2$$

which implies

$$\mathcal{X}_{a[b}\mathcal{X}_{c]d} = \frac{i}{8}\epsilon_{abcd}\mathcal{X}^2 - \frac{1}{2}\mathcal{X}_{ad}\mathcal{X}_{bc} \ .$$

We can evaluate

$$
\begin{aligned}
T_{y[h}\mathcal{H}_{g]x} &= -4\bar{\mathcal{H}}_{yk}\mathcal{H}^k{}_{[h}\mathcal{H}_{g]x} \\
&= -\bar{\mathcal{H}}_{yk}(\frac{i}{2}\epsilon^k{}_{hgx}\mathcal{H}^2 - 2\mathcal{H}^k{}_x\mathcal{H}_{hg}) \\
&= -\frac{1}{2}\mathcal{H}^2(g_{yh}\bar{\mathcal{H}}_{gx} + g_{yg}\bar{\mathcal{H}}_{xh} + g_{yx}\bar{\mathcal{H}}_{hg}) - \frac{1}{2}T_{yx}\mathcal{H}_{hg} \ . \qquad (4.11)
\end{aligned}
$$

This implies

$$
2\bar{\mathcal{H}}^{hk}\mathcal{E}^{dg}{}_{bk} \cong T_b^{[g}\bar{\mathcal{H}}^{h]d}
$$

and

$$
L_b^{gh} \cong \frac{2(q_E + iq_B)\mathcal{C}^{xg}{}_{bk}t_x\bar{\mathcal{H}}^{hk} - \bar{\mathcal{H}}^{hk}\nabla^g\mathcal{B}'_{bk}}{4\bar{\Xi}' - 2M} \ .
$$

Expanding $\mathcal{C}^{dg}{}_{bk}$,

$$
\begin{aligned}
\bar{\mathcal{H}}^{hk}\mathcal{C}^{dg}{}_{bk} &= \bar{\mathcal{H}}^{hk}\mathcal{Q}'^{dg}{}_{bk} + \frac{3P'}{4\bar{\Xi}' - 2M}(\mathcal{F}\tilde{\otimes}\mathcal{F})^{dg}{}_{bk}\bar{\mathcal{H}}^{hk} \\
&= \bar{\mathcal{H}}^{hk}\mathcal{Q}'^{dg}{}_{bk} + \frac{3P'}{(4\bar{\Xi}' - 2M)(4\Xi' - 2M)}(\mathcal{F}\tilde{\otimes}\mathcal{F})^{dg}{}_{bk}((q_E - iqB)\bar{\mathcal{F}}^{hk} - \bar{\mathcal{B}}'^{hk}) \ .
\end{aligned}
$$

Now, considering the anti-symmetric part

$$
\begin{aligned}
(\mathcal{F}\tilde{\otimes}\mathcal{F})^{d[g}{}_{bk}\bar{\mathcal{F}}^{h]k} &= \mathcal{F}^{d[g}\mathcal{F}_{bk}\bar{\mathcal{F}}^{h]k} - \frac{1}{3}\mathcal{F}^2\mathcal{I}^{d[g}{}_{bk}\bar{\mathcal{F}}^{h]k} \\
&= \mathcal{F}^{d[g}\mathcal{F}^{h]k}\bar{\mathcal{F}}_{bk} - \frac{1}{3}\mathcal{F}^2\mathcal{I}^{d[gh]k}\bar{\mathcal{F}}_{bk} \\
&= \frac{i}{8}\mathcal{F}^2\epsilon^{dghk}\bar{\mathcal{F}}_{bk} + \frac{1}{2}\mathcal{F}^{dk}\mathcal{F}^{hg}\bar{\mathcal{F}}_{bk} - \frac{1}{3}\mathcal{F}^2\mathcal{I}^{d[gh]k}\bar{\mathcal{F}}_{bk} \\
&\cong \frac{1}{2}\mathcal{F}^{dk}\mathcal{F}^{hg}\bar{\mathcal{F}}_{bk} + \frac{i}{24}\mathcal{F}^2\epsilon^{dghk}\bar{\mathcal{F}}_{bk} + \frac{1}{12}\mathcal{F}^2 g^{d[h}\bar{\mathcal{F}}^{g]}{}_b \\
&= \frac{1}{2}\mathcal{F}_{bk}\mathcal{F}^{hg}\bar{\mathcal{F}}^{dk} + \frac{1}{6}\mathcal{F}^2\mathcal{I}^{ghdk}\bar{\mathcal{F}}_{bk} \\
&= -\frac{1}{2}(\mathcal{F}\tilde{\otimes}\mathcal{F})^{gh}{}_{bk}\bar{\mathcal{F}}^{dk} \ .
\end{aligned}
$$

So

$$
\begin{aligned}
T_1 = \nabla^a \mathcal{C}_{abcd} = 4\mathcal{I}_{cdgh} L_b^{gh} \\
&= \frac{8(q_E + iq_B)\mathcal{I}_{cdgh}\bar{\mathcal{H}}^{hk}\mathcal{Q}'^{xg}{}_{bk}t_x}{4\bar{\Xi}' - 2M} - \frac{4\mathcal{I}_{cdgh}\bar{\mathcal{H}}^{hk}\nabla^g\mathcal{B}'_{bk}}{4\bar{\Xi}' - 2M} \\
&\quad - \frac{24(q_E + iq_B)P'\mathcal{I}_{cdgh}(\mathcal{F}\tilde{\otimes}\mathcal{F})^{xg}{}_{bk}\bar{\mathcal{B}}'^{hk}t_x}{(4\bar{\Xi}' - 2M)^2(4\Xi' - 2M)} \\
&\quad - \frac{12P'(q_E + iq_B)(q_E - iq_B)}{(4\bar{\Xi}' - 2M)^2(4\Xi' - 2M)}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{cdbk}\bar{\mathcal{F}}^{xk}t_x
\end{aligned}
$$

and we can conclude

$$
\begin{aligned}
T_1 - T_3 = \nabla^a \mathcal{C}_{abcd} - \frac{12P'(q_E + iq_B)\bar{\mathcal{H}}^{xa}t_x}{(4\bar{\Xi}' - 2M)^2}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{abcd} \\
&= \frac{8(q_E + iq_B)\mathcal{I}_{cdgh}\bar{\mathcal{H}}^{hk}\mathcal{Q}'^{xg}{}_{bk}t_x}{4\bar{\Xi}' - 2M} - \frac{4\mathcal{I}_{cdgh}\bar{\mathcal{H}}^{hk}\nabla^g\mathcal{B}'_{bk}}{4\bar{\Xi}' - 2M} \qquad (4.12) \\
&\quad - \frac{24(q_E + iq_B)P'\mathcal{I}_{cdgh}(\mathcal{F}\tilde{\otimes}\mathcal{F})^{xg}{}_{bk}\bar{\mathcal{B}}'^{hk}t_x}{(4\bar{\Xi}' - 2M)^2(4\Xi' - 2M)} \\
&\quad - \frac{12P'(q_E + iq_B)}{(4\bar{\Xi}' - 2M)^2}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{cdbk}\left(\frac{\bar{\mathcal{B}}'^{xk}}{4\Xi' - 2M}\right)t_x \\
&= \frac{q_E + iq_B}{4\bar{\Xi}' - 2M}(\mathcal{A}_1 \circledast \mathcal{Q}')_{bcd} + \frac{1}{4\bar{\Xi}' - 2M}(\mathcal{A}_2 \circledast \nabla\mathcal{B}')_{bcd} \qquad (4.13) \\
&\quad + \frac{(q_E + iq_B)P'}{(4\bar{\Xi}' - 2M)^2(4\Xi' - 2M)}(\mathcal{A}_3 \circledast \mathcal{B}')_{bcd}
\end{aligned}
$$

as desired.

For T_4, we have

$$
\mathcal{E}^{xa}{}_{ab} = -\frac{1}{2}T_b^x \ .
$$

For T_5,

$$
\mathcal{F}_{ab}\nabla^a\mathcal{F}_{cd} = 2(\mathcal{C}^{xa}{}_{cd}\mathcal{F}_{ab} + \mathcal{E}^{xa}{}_{cd}\mathcal{F}_{ab})t_x
$$

Now,

$$\begin{aligned}\mathcal{E}^{xa}{}_{cd}\mathcal{F}_{ab} &= -\frac{1}{2}(g \owedge T)_{lmcd}\bar{\mathcal{I}}^{xalm}\mathcal{F}_{ba}\\ &= -\frac{1}{4}(g_{lc}T_{md} + g_{md}T_{lc})(g^{lx}\mathcal{F}_b{}^m - g^{mx}\mathcal{F}_b{}^l - g^l_b\mathcal{F}^{mx} + g^m_b\mathcal{F}^{lx} + g^x_b\mathcal{F}^{ml})\\ &= -\frac{1}{2}(g^x_{[c}T_{d]m}\mathcal{F}_b{}^m - T^x_{[c}\mathcal{F}_{d]b} - g_{b[c}T_{d]m}\mathcal{F}^{mx} - T_{b[c}\mathcal{F}_{d]}{}^x + g^x_b\mathcal{F}^m{}_{[c}T_{d]m})\ .\end{aligned}$$

Rewriting $T_{xy} = \frac{4}{4\bar{\Xi}'-2M}\bar{\mathcal{H}}_x{}^z((q_E + iq_B)\mathcal{F}_{yz} - \mathcal{B}'_{yz})$,

$$\begin{aligned}-2(4\bar{\Xi}' - 2M)\mathcal{E}^{xa}{}_{cd}\mathcal{F}_{ab} &= (\mathcal{A}_1 \circledast \mathcal{B}')^x{}_{bcd} - g^x_{[c}\bar{\mathcal{H}}_{d]b}(q_E + iq_B)\mathcal{F}^2\\ &\qquad - g_{b[c}\bar{\mathcal{H}}_{d]x}(q_E + iq_B)\mathcal{F}^2 + g^x_b\bar{\mathcal{H}}_{dc}(q_E + iq_B)\mathcal{F}^2\\ &\qquad + 4\bar{\mathcal{H}}^{xk}(q_E + iq_B)\mathcal{F}_{k[c}\mathcal{F}_{d]b} + 4\bar{\mathcal{H}}_b{}^k(q_E + iq_B)\mathcal{F}_{k[c}\mathcal{F}_{d]}{}^x\\ &= (\mathcal{A}_1 \circledast \mathcal{B}')^x{}_{bcd} + 2(q_E + iq_B)(\bar{\mathcal{H}}^{xk}\mathcal{F}_{bk} + \bar{\mathcal{H}}_{bk}\mathcal{F}^{xk})\mathcal{F}_{cd}\\ &= (\mathcal{A}_1 \circledast \mathcal{B}')^x{}_{bcd} + (4\bar{\Xi}' - 2M)T^x_b\mathcal{F}_{cd}\end{aligned}$$

and

$$\mathcal{E}^{xa}{}_{cd}\mathcal{F}_{ab} = -\frac{1}{2}T^x_b\mathcal{F}_{cd} + \frac{1}{4\bar{\Xi}' - 2M}(\mathcal{A}_1 \circledast \mathcal{B}')^x{}_{bcd}\ . \tag{4.14}$$

On the other hand

$$\mathcal{C}^{xa}{}_{cd}\mathcal{F}_{ab} = \mathcal{Q}'^{xa}{}_{cd}\mathcal{F}_{ab} + \frac{3P'}{4\bar{\Xi}' - 2M}(\mathcal{F}\tilde{\otimes}\mathcal{F})^{xa}{}_{cd}\mathcal{F}_{ab}$$

the second term of which we write

$$(\mathcal{F}\tilde{\otimes}\mathcal{F})^{xa}{}_{cd}\mathcal{F}_{ab} = -\mathcal{F}^2 g^x_b\mathcal{F}_{cd} - \frac{1}{3}\mathcal{F}^2\mathcal{I}^{xa}{}_{cd}\mathcal{F}_{ab}\ .$$

For T_6,

$$\begin{aligned}\mathcal{F}^{ef}\nabla^a\mathcal{F}_{ef} &= 2\mathcal{C}^{xa}{}_{ef}\mathcal{F}^{ef}t_x + 2\mathcal{E}^{xa}{}_{ef}\mathcal{F}^{ef}t_x \\ &= 2\mathcal{Q}'^{xa}{}_{ef}t_x\mathcal{F}^{ef} + \frac{4P'\mathcal{F}^2}{4\bar{\Xi}' - 2M}\mathcal{F}^{xa} - \frac{4(q_E + iq_B)(4\bar{\Xi}' - 2M)}{P'^4}\bar{\mathcal{H}}^{xa} \ .\end{aligned}$$

Lastly, observe that for T_3,

$$\frac{q_E + iq_B}{4\bar{\Xi}' - 2M}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{abcd}\bar{\mathcal{H}}^{xa} = \frac{1}{4\bar{\Xi}' - 2M}(\mathcal{A}_1 \circledast \mathcal{B}')^x{}_{bcd} - T^x_b\mathcal{F}_{cd} - \frac{q_E + iq_B}{3(4\bar{\Xi}' - 2M)}\mathcal{F}^2\mathcal{I}_{abcd}\bar{\mathcal{H}}^{xa}$$

so that after a bit of simple regrouping of terms,

$$T_2 + 2T_3 - \frac{3P'}{4\bar{\Xi}' - 2M}(T_4 + T_5 + T_6) = \upsilon_1\mathcal{A}_1 \circledast \mathcal{Q}' + \upsilon_2\mathcal{A}_2 \circledast \mathcal{B}'$$

as desired, where υ_* depends only on P' and $(4\bar{\Xi} - 2M)^{-1}$. Combining this with (4.13), we obtain the desired result. □

In view of the previous two propositions, if the vector $\mathcal{S} = (\mathcal{B}'_{ab}, \nabla_c\mathcal{B}'_{ab}, \mathcal{Q}'_{abcd})$ is defined to be a smooth section in $T^0_2\mathcal{M} \oplus T^0_3\mathcal{M} \oplus T^0_4\mathcal{M}$, then $\mathcal{S}$ satisfies a wave equation of type (4.1), namely

$$\Box_g\mathcal{S} = (\upsilon_1\mathcal{A}_1) \circledast \mathcal{S} + (\upsilon_2\mathcal{A}_2) \circledast \nabla\mathcal{S} \tag{4.15}$$

where $(\upsilon_*\mathcal{A}_*)$ stands for matrices representing linear transformations on $T^0_2\mathcal{M} \oplus T^0_3\mathcal{M} \oplus T^0_4\mathcal{M}$ with entries consisting of smooth geometric tensors multiplied by coefficients depending on P' and $(4\bar{\Xi}' - 2M)^{-1}$. It is to this $\mathcal{S}$ that we will apply the Carleman estimates.

4.2 Initial value on the bifurcate horizon

In this section, the geometric constraint of a non-expanding bifurcate event horizon and some technical assumptions prescribed at the bifurcate sphere will be seen to together imply the vanishing of $\mathcal{B}_{ab}$ and $\mathcal{Q}_{abcd}$ on the event horizon. We will use the notation $\Psi_i^{(\mathcal{K})}$ for the complex scalars associated to the Weyl-field $\mathcal{K}_{abcd}$ in a null tetrad, similarly the notation $\Upsilon_i^{(\mathcal{G})}$ for complex scalars associated to a two form $\mathcal{G}_{ab}$. See Section 2.4 for the definition of the scalars in terms of tetrad directions. Notice that we have $\mathcal{G}^2 = -4(\Upsilon_0^{(\mathcal{G})})^2 + 4\Upsilon_1^{(\mathcal{G})}\Upsilon_{-1}^{(\mathcal{G})}$.

Recall from Section 2.5 that we assume our space-time admits a smooth bifurcate horizon which is non-expanding (see also Section 1.3 for justification). We were able to conclude that $\Upsilon_1^{(\mathcal{H})} = \Psi_2^{(\mathcal{C})} = \Psi_1^{(\mathcal{C})} = 0$ on $\mathfrak{H}^+$, and that $\Psi_0^{(\mathcal{C})}$ and $\Upsilon_0^{(\mathcal{H})}$ are constant along the generators of $\mathfrak{H}^\pm$.

Now first consider $\mathcal{F}_{ab}$. Since $\Upsilon_1^{(\mathcal{F})} = -F(m,l) = -g(\nabla_m t, l)$, and $t \in T\mathfrak{H}^+$, we have $g(t,l) = 0$. Therefore we have

$$\Upsilon_1^{(\mathcal{F})} = g(t, \nabla_m l) = 0 \ , \ \text{on } \mathfrak{H}^+$$

by the vanishing of θ and ϑ. Similarly $\Upsilon_{-1}^{(\mathcal{F})} = 0$ on $\mathfrak{H}^-$. Therefore by the same argument as in Section 2.5, $\mathcal{B}'_{ab}$ is a free Maxwell field such that $\Upsilon_{\pm 1}^{(\mathcal{B}')}|_{\mathfrak{H}^\pm} = 0$, and thus $\Upsilon_0^{(\mathcal{B}')}$ is constant on the geodesic generators of the event horizon. Now, if we assume that $\Upsilon_0^{(\mathcal{B}')}$ vanishes on the event horizon, we see that the Maxwell equation (2.24b) implies $(D - \Gamma_{124})\Upsilon_{-1}^{(\mathcal{B}')} = 0$ (similarly for Υ_{-1}), and therefore $\Upsilon_{\pm 1}^{(\mathcal{B}')}$ are constant along generators of the event horizon. We have thus demonstrated

Lemma 4.2.1. *Let $(\mathcal{M}, g_{ab}, H_{ab})$ be a stationary asymptotically flat solution to the Einstein-Maxwell system. Assume the space-time admits a smooth non-expanding bifurcate event horizon $\mathfrak{H}^\pm$. Define Ξ' as in Theorem 3.5.1. Then a sufficient condition for $\mathcal{B}'_{ab}$ as defined in (3.19a) to vanish identically on the bifurcate horizon is $\Upsilon_0^{(\mathcal{B}')} = 0$*

and that $|q_E + iq_B|^2/P' = 2\Xi'$ on the bifurcate sphere.

We will argue in a similar fashion for $\mathcal{Q}'_{abcd}$. Consider the Weyl tensor $(\mathcal{F}\tilde{\otimes}\mathcal{F})_{abcd}$. It is easily verified that[1]

$$\begin{aligned}
\Psi_{\pm 2}^{(\mathcal{F}\tilde{\otimes}\mathcal{F})} &= (\Upsilon_{\pm 1}^{(\mathcal{F})})^2 \\
\Psi_{\pm 1}^{(\mathcal{F}\tilde{\otimes}\mathcal{F})} &= -\Upsilon_{\pm 1}^{(\mathcal{F})}\Upsilon_0^{(\mathcal{F})} \\
\Psi_0^{(\mathcal{F}\tilde{\otimes}\mathcal{F})} &= -\frac{1}{3}[\Upsilon_1^{(\mathcal{F})}\Upsilon_{-1}^{(\mathcal{F})} + 2(\Upsilon_0^{(\mathcal{F})})^2]
\end{aligned}$$

In particular, $\Psi_{\pm 2}^{(\mathcal{F}\tilde{\otimes}\mathcal{F})} = \Psi_{\pm 1}^{(\mathcal{F}\tilde{\otimes}\mathcal{F})} = 0$ on $\mathfrak{H}^{\pm}$. It therefore suffices to consider $\Psi_{-2}^{(\mathcal{Q}')}$, $\Psi_{-1}^{(\mathcal{Q}')}$, and $\Psi_0^{(\mathcal{Q}')}$ on $\mathfrak{H}^+$.

Now assume the conditions for Lemma 4.2.1 is satisfied, so we can assume $\mathcal{B}'_{ab} = 0$ on the bifurcate event horizon.

Lemma 4.2.2. *Assuming the basic geometric set-up as in Lemma 4.2.1. Also assume the conditions are satisfied such that $\mathcal{B}'_{ab} = 0$ on $\mathfrak{H}^{\pm}$. Then $\Psi_0^{(\mathcal{Q}')}$ is constant along the generators of the horizon.*

Proof. We separately consider the case of $q_E + iq_B = 0$ and $q_E + iq_B \neq 0$. As discussed in Section 3.5, in the case of vanishing charge, $\Xi' = 0$ by definition, and the assumption that $\mathcal{B}'_{ab} = 0$ leads to that $\mathcal{H}_{ab} = 0$ on the horizon. This implies that $\mathcal{F}_{ab}$ obeys a source-free Maxwell equation when restricted to the horizon, and in particular $\Upsilon_0^{(\mathcal{F})}$ is constant along generators of the horizon, and thus also P'. By definition of $\mathcal{Q}'_{abcd}$, if follows that $\Psi_0^{(\mathcal{Q}')}$ is constant along generators of the horizon.

In the case of non-vanishing charge. The vanishing of $\mathcal{B}'_{ab}$ allows us to re-write

$$\frac{3P'}{4\bar{\Xi}' - 2M}\Psi_0^{(\mathcal{F}\tilde{\otimes}\mathcal{F})} = -\frac{P'(4\bar{\Xi}' - 2M)}{(q_E + iq_B)^2}(\Upsilon_0^{(\mathcal{H})})^2 \ .$$

[1]Indeed, this is one of the reasons behind the definition of the symmetric spinor product to start with.

Noticing that $\Upsilon_0^{(\mathcal{H})}$ is constant along the horizon, it suffices to consider DP' and $D\Xi'$ on $\mathfrak{H}^+$. In the latter, $D\Xi' = (q_E - iq_B)\mathcal{H}(t,l)$. Using that t is tangent to the horizon, immediately $D\Xi' = 0$. Similarly, in the former, using (4.5), we see that $DP' \propto \mathcal{C}_{abcd}t^a l^b \mathcal{F}^{cd} \propto \Psi_0^{(\mathcal{C})}\Upsilon_1^{(\mathcal{F})} + \Psi_1^{(\mathcal{C})}\Upsilon_0^{(\mathcal{F})} + \Psi_2^{(\mathcal{C})}\Upsilon_{-1}^{(\mathcal{F})}$ and hence $DP' = 0$. Therefore we conclude that $\Psi_0^{(\mathcal{Q}')}$ is constant along generators of the horizon. □

With this in mind, using the reduced Bianchi identities (2.27a, 2.27b), we can also prove a criterion for the vanishing of $\mathcal{Q}'_{abcd}$ on $\mathfrak{H}^\pm$.

Lemma 4.2.3. *Let $(\mathcal{M}, g_{ab}, H_{ab})$ be a stationary asymptotically flat solution to the Einstein-Maxwell system. Assume the space-time admits a smooth non-expanding bifurcate event horizon $\mathfrak{H}^\pm$. Define Ξ' as in Theorem 3.5.1. Further assume that $\mathcal{B}'_{ab}$ vanishes on $\mathfrak{H}^\pm$. Then a sufficient condition for $\mathcal{Q}'_{abcd}$ as defined in (3.19b) to vanish identically on $\mathfrak{H}^\pm$ is that $\Psi_0^{(\mathcal{Q}')} = 0$ on the bifurcate sphere $\mathfrak{H}_0$.*

Proof. Recall that $\Psi_{-2}^{(\mathcal{Q}')}$ and $\Psi_{-1}^{(\mathcal{Q}')}$ vanishes on $\mathfrak{H}_0$. Therefore in suffices to demonstrate that their D derivative is proportional to themselves. Again we treat the cases with and without charge separately.

Assume first that the charge vanishes. By the same argument as in the proof of Lemma 4.2.2, we have that $\mathcal{H}_{ab}$ vanishes on the horizon and $\mathcal{F}_{ab}$ is free Maxwell on the horizon. Now consider

$$
\begin{aligned}
(D - \Gamma_{124})\left(\frac{3P'}{4\bar{\Xi}' - 2M}\Psi_{-1}^{(\mathcal{F}\tilde{\otimes}\mathcal{F})}\right) &= -\frac{3P'}{4\bar{\Xi}' - 2M}\Upsilon_0^{(\mathcal{F})}(D - \Gamma_{124})\Upsilon_{-1}^{(\mathcal{F})} \\
&= \frac{3P'}{4\bar{\Xi}' - 2M}\Upsilon_0^{(\mathcal{F})}(\bar{\delta} + 2\underline{\eta})\Upsilon_0^{(\mathcal{F})} \\
&= \frac{3P'}{4\bar{\Xi}' - 2M}\frac{1}{2}(\bar{\delta} + 4\underline{\eta})(\Upsilon_0^{(\mathcal{F})})^2 \ .
\end{aligned}
$$

Observe that $(P')^4 = -(4\bar{\Xi}' - 2M)^2/\mathcal{F}^2 = (4\bar{\Xi}' - 2M)^2/(2\Upsilon_0^{(\mathcal{F})})^2$. Immediately

$$
\nabla P' = -\frac{P'}{4(\Upsilon_0^{(\mathcal{F})})^2}\nabla(\Upsilon_0^{(\mathcal{F})})^2 \Rightarrow \nabla(P'(\Upsilon_0^{(\mathcal{F})})^2) = \frac{3}{4}P'\nabla(\Upsilon_0^{(\mathcal{F})})^2
$$

if the derivative ∇ is taken in $T\mathfrak{H}^+$. This implies

$$(D-\Gamma_{124})(\frac{3P'}{4\bar{\Xi}'-2M}\Psi_{-1}^{(\mathcal{F}\tilde{\otimes}\mathcal{F})}) = (\bar{\delta}+3\underline{\bar{\eta}})\frac{2P'}{4\bar{\Xi}'-2M}(\Upsilon_0^{(\mathcal{F})})^2$$
$$= -(\bar{\delta}+3\underline{\bar{\eta}})(\frac{3P'}{4\bar{\Xi}'-2M}\Psi_0^{(\mathcal{F}\tilde{\otimes}\mathcal{F})})$$

which, in view of (2.27b) and initial assumptions in the statement of the lemma, means that $(D-\Gamma_{124})\Psi_{-1}^{(\mathcal{Q}')}=0$, and hence $\Psi_{-1}^{(\mathcal{Q}')}=0$ on $\mathfrak{H}^+$.

For $\Psi_{-2}^{(\mathcal{Q}')}$, it suffices to observe that Proposition 4.1.2 implies that $\mathcal{Q}'_{abcd}$ is a Weyl field whose source depends on itself, $\mathcal{B}'_{ab}$ and $\nabla_c\mathcal{B}'_{ab}$. Therefore it satisfies an equation similar to (2.27a) with additional source terms (and no terms coming from the Ricci tensor). By the vanishing of all other components of $\Psi_*^{(\mathcal{Q}')}$, and the vanishing of $\mathcal{B}'_{ab}$, it is clear that $\Psi_{-2}^{(\mathcal{Q}')}$ satisfies a transport equation of the form

$$D\Psi_{-2} = A\Psi_{-2}$$

where we can remove the dependence on the $\nabla\mathcal{B}'_{ab}$ term because, as seen in (4.13), that term is also linear in $\mathcal{H}_{ab}$, which is seen to vanish on the horizon in the case without charge.

Now consider the case where the charge does not vanish. By the identification of $(q_E+iq_B)\mathcal{F}_{ab} = (4\bar{\Xi}'-2M)\mathcal{H}_{ab}$, a similar computation can be performed. First remark that as in the proof of Lemma 4.2.2, $DP' = D\Xi' = 0$.

$$(D-\Gamma_{124})(\frac{3P'}{4\bar{\Xi}'-2M}\Psi_{-1}^{(\mathcal{F}\tilde{\otimes}\mathcal{F})}) = (D-\Gamma_{124})(\frac{3P'(4\bar{\Xi}'-2M)}{(q_E+iq_B)^2}\Psi_{-1}^{(\mathcal{H}\tilde{\otimes}\mathcal{H})})$$
$$= -\frac{3P'(4\bar{\Xi}'-2M)}{(q_E+iq_B)^2}\Upsilon_0^{(\mathcal{H})}(D-\Gamma_{124})\Upsilon_{-1}^{(\mathcal{H})}$$
$$= \frac{3P'(4\bar{\Xi}'-2M)}{2(q_E+iq_B)^2}(\bar{\delta}+4\underline{\bar{\eta}})(\Upsilon_0^{(\mathcal{H})})^2 \ .$$

Now note that $(P')^{-4} = -\mathcal{H}^2/(q_E+iq_B)^2$ under the assumption of $\mathcal{B}'_{ab}=0$. So just as before

$$(D-\Gamma_{124})(\frac{3P'}{4\bar{\Xi}'-2M}\Psi_{-1}^{(\mathcal{F}\tilde{\otimes}\mathcal{F})}) = (4\bar{\Xi}'-2M)(\bar{\delta}+3\underline{\eta})\frac{2P'}{(q_E+iq_B)^2}(\Upsilon_0^{(\mathcal{H})})^2 \ .$$

Now, let us re-examine Proposition 3.3.2. In it, to demonstrate that $\nabla_c(P^{-1}-2\Xi)=0$, we need that $\mathcal{Q}_{dcab}\mathcal{H}^{ab}t^d=0$. For the proposition we assumed that $\mathcal{Q}_{abcd}$ vanishes in all components, but in individual situations, it will not be strictly necessary! For example, consider $2\delta\Xi'$ on $\mathfrak{H}^+$. Since the $\mathcal{H}(l,m)$ component vanishes, the only components of $\mathcal{Q}_{dcab}\mathcal{H}^{ab}t^d$ to consider are $\mathcal{Q}(t,m,\underline{l},l)$ and $\mathcal{Q}(t,m,m,l)$, both of which are already known to vanish on $\mathfrak{H}^+$. Hence on $\mathfrak{H}^+$ $2\delta\Xi'=|q_E+iq_B|^2\delta(P')^{-1}$. Using this fact

$$4\bar{\delta}\bar{\Xi}' = -\frac{2|q_E-iq_B|^2}{(\bar{P}')^2}\bar{\delta}\bar{P}' = \frac{|q_E-iq_B|^2}{4\bar{P}'(\bar{\Upsilon}_0^{(\mathcal{H})})^2}2\bar{\delta}(\bar{\Upsilon}_0^{(\mathcal{H})})^2 = 2(q_E+iq_B)\bar{P}'\bar{\delta}(\bar{\Upsilon}_0^{(\mathcal{H})}) \ .$$

so

$$(D-\Gamma_{124})(\frac{3P'\Psi_{-1}^{(\mathcal{F}\tilde{\otimes}\mathcal{F})}}{4\bar{\Xi}'-2M}) = -(\bar{\delta}+3\underline{\eta})(\frac{3P'\Psi_{0}^{(\mathcal{F}\tilde{\otimes}\mathcal{F})}}{4\bar{\Xi}'-2M}) - \frac{2\bar{P}'}{P'}\Upsilon_0^{(\mathcal{H})}\bar{\delta}(\bar{\Upsilon}_0^{(\mathcal{H})}) \ .$$

Observe that

$$\delta P'\bar{\delta}P' = \frac{(P')^4}{|q_E+iq_B|^4}4\delta\Xi'\bar{\delta}\Xi' = -\frac{4}{\mathcal{H}^2}\mathcal{H}(t,m)\mathcal{H}(t,\bar{m})$$

Since $\mathcal{H}(t,l)=0$, $8\mathcal{H}(t,m)\mathcal{H}(t,\bar{m})=\mathcal{H}^2t^2$. So this implies that $\delta P'\bar{\delta}P'\in\mathbb{R}_-$. In other words $\bar{\delta}\bar{P}'=-\bar{\delta}P'$. A little bit of algebraic manipulations gives

$$\begin{aligned}
-2\frac{\bar{P}'}{P'}\Upsilon_0^{(\mathcal{H})}\bar{\delta}(\bar{\Upsilon}_0^{(\mathcal{H})}) &= -2\frac{|q_E+iq_B|^2\bar{P}'}{P'}\frac{1}{P'^2}\bar{\delta}\frac{1}{\bar{P}'^2} \\
&= 4|q_E+iq_B|^2\frac{1}{P'^3}\frac{1}{\bar{P}'^2}\bar{\delta}\bar{P}' \\
&= 2|q_E+iq_B|^2\frac{1}{\bar{P}'^2}\bar{\delta}\frac{1}{P'^2} \\
&= 2\bar{\Upsilon}_0^{(\mathcal{H})}\bar{\delta}\Upsilon_0^{(\mathcal{H})} = \bar{\delta}\frac{1}{2}\Phi_0 - 2\Upsilon_0^{(\mathcal{H})}\bar{\delta}\bar{\Upsilon}_0^{(\mathcal{H})}
\end{aligned}$$

to which we now apply the Maxwell equation (2.24b) and compare to the reduced Bianchi identity (2.27b) to see that $(D - \Gamma_{124})\Psi_{-1}^{(\mathcal{Q}')} = 0$.

For the term $\Psi_{-2}^{(\mathcal{Q}')}$, we argue the same as in the charge-less case. But now we need to control (by examining the divergence relation) $\mathcal{I}_{cdgh}\bar{\mathcal{H}}^{hk}\nabla^g\mathcal{B}'_{bk}\bar{m}^b\underline{l}^c\bar{m}^d$ since $\mathcal{H}_{ab}$ no longer vanishes on the boundary and $\nabla_c\mathcal{B}'_{ab}$ is not a priori controllable (may contain derivatives transversal to the null hypersurface). A closer examination reveals, however, that the only non-zero terms from $\mathcal{I}_{cdgh}$ is $\mathcal{I}(\underline{l}, \bar{m}, l, m)$, so the only term carrying transversal derivatives of $\mathcal{B}'_{ab}$ is

$$tr\,\mathcal{I}(\underline{l}, \bar{m}, l, m)\bar{\mathcal{H}}(\bar{m}, \cdot)\underline{D}\mathcal{B}'(\bar{m}, \cdot) \ .$$

By inserting the proper pairs from the null tetrad to compute the trace, it is immediately obvious that the only term for which the $\bar{\mathcal{H}}$ term does not vanish is $\bar{\mathcal{H}}(\bar{m}, m)$, whose pairing requires us to evaluate $\underline{D}\mathcal{B}'(\bar{m}, \bar{m})$ which is always zero as $\mathcal{B}'$ is constructed to be antisymmetric. Hence $\Psi_{-2}^{(\mathcal{Q})'}$ also satisfies a transport equation of type $D\Psi_{-2} = A\Psi_{-2}$, which by the condition that $\Psi_{-2}^{(\mathcal{Q}')}$ vanishes on the bifurcate sphere, implies that $\Psi_{-2}^{(\mathcal{Q}')} = 0$ on the horizon. □

Remark 4.2.4. *One can also in principle demonstrate the vanishing of the* $\Psi_{-2}^{(\mathcal{Q}')}$ *components by considering the Bianchi identity (2.27a). The computations, however, is not any more enlightening then the argument given herein.*

Now we give an example of sufficient scalar conditions (these are what we will use in the sequel) for $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ to vanish on the horizon.

Corollary 4.2.5. *Let* $(\mathcal{M}, g_{ab}, H_{ab})$ *be a stationary asymptotically flat solution to the Einstein-Maxwell system. Assume the space-time admits a smooth non-expanding bifurcate event horizon* $\mathfrak{H}^\pm$*, and assume that the stationary Killing vector field* t *only vanishes on a discrete subset of* $\mathfrak{H}_0$*. Define* Ξ' *as in Theorem 3.5.1. If we further assume that on the bifurcate sphere* $\mathfrak{H}_0$*:*

- $4|\Xi'| < 2M$
- $\mathcal{B}'^2 = 0$, *and*
- $(4\bar{\Xi}' - 2M)\partial_c(P')^{-1} = 2\mathcal{F}_{ac}t^a$, *when the derivative is taken in directions tangent to* $\mathfrak{H}_0$.

then $\mathcal{Q}'_{abcd}$ *and* $\mathcal{B}'_{ab}$ *vanishes on* $\mathfrak{H}^{\pm}$.

Proof. The first condition is necessary in guaranteeing that the quantities which depend on $(4\bar{\Xi}' - 2M)^{-1}$ are well-defined. Notice that it is trivially satisfied in the case with vanishing charge.

Observe that due to the vanishing of the $\Upsilon^{(\mathcal{B}')}_{\pm 1}$, $\mathcal{B}'^2 = -4(\Upsilon^{(\mathcal{B}')}_0)^2$ and immediately the second condition implies that $\Upsilon^{(\mathcal{B}')}_0 = 0$ on $\mathfrak{H}_0$.

A quick computation analogous to that for (4.5) shows that the third equality implies

$$\frac{(4\bar{\Xi}' - 2M)}{P'}\mathcal{C}_{dcab}t^d\mathcal{F}^{ab} = 2\mathcal{F}^2\mathcal{F}_{dc}t^d \ .$$

Now, noticing that t is tangent to $\mathfrak{H}_0$, and that the only non-vanishing component of $\mathcal{F}_{ab}$ on the bifurcate sphere is $\Upsilon^{(\mathcal{F})}_0$, we get

$$\frac{(4\bar{\Xi}' - 2M)}{P'}g(t,m)\Psi^{(\mathcal{C})}_0 = 2(\Upsilon^{(\mathcal{F})}_0)^2 g(t,m)$$

which, by the assumption that t only vanishes discretely, gives $\frac{(4\bar{\Xi}'-2M)}{P'}\Psi^{(\mathcal{Q}')}_0 = 0$. In particular, by the first condition assumed in the corollary, $\Psi^{(\mathcal{Q}')}_0 = 0$.

By Lemmas 4.2.1 and 4.2.3 we have the desired result. □

Remark 4.2.6. *In the case with non-vanishing charge, the third condition can be replaced by requiring* $|q_E + iq_B|^2/P' = 2\Xi'$, *a form more consistent with the conditions given in [14].*

Remark 4.2.7. *The first condition in the Corollary, that* $|4\bar{\Xi}'| < 2M$, *is consistent with the physical assumption that the charge of the space-time is smaller than the* *mass. Indeed, as we have seen in Chapter 3, the condition can be heuristically re-written as $|q_E + iq_B|^2/|P'| < M$. In the Reissner-Nordström case, P' was seen to be the radius r. At the event horizon, $|q_E + iq_B|^2/P' - M = -\sqrt{M^2 - |q_E + iq_B|^2} < 0$. This also illustrates why the condition is not necessary when the charge vanishes.*

4.3 Carleman estimate

In the proof we will use the generalized Carleman estimates due to Ionescu and Klainerman [14]. Here we collect the statements and definitions.

Let B_1 be the open ball of radius 1 centered at the origin in Minkowski space. Assume we have a coördinate chart Π^{x_0} from B_1 to some neighborhood of $x_0 \in \mathcal{M}$, with the image of the origin being x_0. Denote the image $\Pi^{x_0}(B_1) =: B(x_0)$. By abuse of notation, write g also for the pull-back of the metric $(\Pi^{x_0}_* g)$ on B_1. (In the following, we will often abuse the diffeomorphism Π^{x_0} that, functions and tensors defined on the neighborhood $B(x_0)$ will be identified with their pull-backs to B_1 and vice versa.) Let $B \subset B_1$ be an open set, and let ϕ be a complex-valued smooth function on B. Let j be a non-negative integer, we will write

$$|\partial^{(j)}\phi(x)| := \sum_{\alpha_1,\ldots,\alpha_j=1}^{4} |\frac{\partial}{\partial x^{\alpha_1}} \cdots \frac{\partial}{\partial x^{\alpha_j}} \phi(x)| \tag{4.16}$$

for the size of the j-th derivative of ϕ. This is necessary as the geometric norm is Lorentzian and is not positive definite.

Let V be a smooth vector field on B_1. Expressed in coördinates, we can write $V = \sum_{\alpha=1}^{4} V^\alpha \frac{\partial}{\partial x^\alpha}$. We assume there exists a number $A_0(V)$ such that

$$\sup_{B_1} \sum_{j=0}^{4} \sum_{\beta=1}^{4} |\partial^{(j)} V^\beta| \leq A_0 \ , \tag{4.17}$$

in other words we control the norms of the first 4 derivatives of the coefficients of V.

Definition 4.3.1. *Fix* $0 < \epsilon_1 \leq 1/A_0(V)$*. A family of weights* $h_\epsilon : B_{\epsilon^{10}} \to \mathbb{R}_+$ *defined for* $0 < \epsilon < \epsilon_1$ *is called* V-conditional pseudo-convex *if for any* $\epsilon \in (0, \epsilon_1)$ *the following are satisfied:*

$$h_\epsilon(0) = \epsilon \ , \quad \sup_{B_{\epsilon^{10}}} \sum_{j=1}^{4} \epsilon^j |\partial^{(j)} h_\epsilon(x)| \leq \epsilon/\epsilon_1 \ , \quad |V(h_\epsilon)(0)| \leq \epsilon^{10} \ . \tag{4.18a}$$

Writing ∇ *for the metric connection, and taking contractions relative to the metric* g

$$\nabla^a h_\epsilon \nabla^b h_\epsilon (\nabla_a h_\epsilon \nabla_b h_\epsilon - \epsilon \nabla^2_{ab} h_\epsilon)|_{x=0} \geq \epsilon_1^2 \ . \tag{4.18b}$$

And $\exists \mu \in [-\epsilon_1^{-1}, \epsilon_1^{-1}]$ *such that for any tangent vector at* $T_0 B_1$, $\sum_{\alpha=1}^{4} X^\alpha \frac{\partial}{\partial x^\alpha}$,

$$\epsilon_1^2 \sum_{\alpha=1}^{4} (X^\alpha)^2 \leq \mu g(X, X) - \nabla^2_{X,X} h_\epsilon + \epsilon^{-2}(|g(X, V)|^2 + |\nabla_X h_\epsilon|^2)|_{x=0} \ . \tag{4.18c}$$

Definition 4.3.2. *A function* $e_\epsilon : B_{\epsilon^{10}} \to \mathbb{R}$ *will be called a* negligible perturbation *if*

$$\sup_{B_{\epsilon^{10}}} |\partial^{(j)} e_\epsilon| \leq \epsilon^{10} \tag{4.19}$$

for $j = 0, \ldots, 4$.

For justification of the pseudo-convexity condition given, see Remark 3.2 in [14]. With the above definitions, we have the following Carleman inequality.

Proposition 4.3.3 (Ionescu-Klainerman [14])**.** *Fix the vector field* V *and the constant* $A_0(V)$*. Fix* ϵ_1 *as in Definition 4.3.1, and let* $\{h_\epsilon\}$ *be a* V*-conditional pseudo-convex family of weights, and* $\{e_\epsilon\}$ *a family of negligible perturbations. Then there is a* $\epsilon \in (0, \epsilon_1)$ *sufficiently small and a constant* $\tilde{C}_\epsilon$ *sufficiently large that for any* $\lambda \geq \tilde{C}_\epsilon$ *and any* $\phi \in C_0^\infty(B_{\epsilon^{10}})$,

$$\lambda \| e^{-\lambda f_\epsilon} \phi \|_{L^2} + \| e^{-\lambda f_\epsilon} |\partial^{(1)} \phi| \|_{L^2} \leq \tilde{C}_\epsilon \lambda^{-1/2} \| e^{-\lambda f_\epsilon} \Box_g \phi \|_{L^2} + \epsilon^{-6} \| e^{-\lambda f_\epsilon} V(\phi) \|_{L^2} \ , \tag{4.20}$$

where $f_\epsilon := \ln(h_\epsilon + e_\epsilon)$.

4.4 Uniqueness of Kerr-Newman metric

In this section, we give a proof of the conditional uniqueness of the Kerr-Newman metric among smooth stationary asymptotically flat solutions to the Einstein-Maxwell equations. First we give the assumptions. Let $(\mathcal{M}, g_{ab}, H_{ab})$ be a smooth space-time solving the Einstein-Maxwell equations. Let t be a smooth Killing vector-field.

(AF) We assume the solution and the vector field t is stationary asymptotically flat. See Remark 3.1.1. We can rephrase the decay condition (slightly strengthened) here, following [14]. Let $\mathcal{M}^\infty$ be the stationary asymptotic end diffeomorphic to $\mathbb{R} \times (\mathbb{R}^3 \setminus B_R)$ for some large radius R. Assume that in the local coördinates (s, x^1, x^2, x^3) given by this diffeomorphism, we have $\partial_s = t$, and that with $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$,

$$g(t,t) = -1 + \frac{2M}{r} + O(r^{-2}) \ , \quad g(x^\alpha, x^\beta) = \delta_{\alpha\beta} + O(r^{-1}) \ , \quad g(t, x^\alpha) = O(r^{-2}) \tag{4.21}$$

for some $M > 0$.

We define the black hole, white hole, and exterior regions $\mathfrak{B}, \mathfrak{W}, \mathfrak{D}$ as in Section 1.3. We also assume that there exists an embedded space-like hypersurface Σ_0 diffeomorphic to $\mathbb{R}^3 \setminus B_{1/2}$. We ask that $\Sigma_0 \cap \mathcal{M}^\infty$ is equal to the $s = 0$ slice. Denote by T_0 the future directed unit normal to Σ_0. Assume that every orbit of t in $\mathfrak{D}$ is complete and intersects $\Sigma_0 \cap \mathfrak{D}$ transversely.

(SBS) Let $\mathfrak{H}^\pm$ and $\mathfrak{H}_0$ be defined as before. Assume that $\mathfrak{H}_0 \subset \Sigma_0$ and is equal to the image of the sphere of radius 1 in $\mathbb{R}^3 \setminus B_{1/2}$ under the diffeomorphism given above.

Also assume that there exists a neighborhood O of $\mathfrak{H}_0$ such that $\mathfrak{H}^\pm \cap O$ are are smooth null hypersurfaces that are non-expanding and intersect transversally in $\mathfrak{H}_0$. We will abuse the notation to also denote $\mathfrak{H}^\pm \cap O$ by $\mathfrak{H}^\pm$ where no confusion is possible. Also assume that t is tangent to $\mathfrak{H}^\pm$ and does not vanish identically on $\mathfrak{H}_0$.

(T) We also need some technical conditions on $\mathfrak{H}_0$. Define Ξ' as in Theorem 3.5.1, and $P', \mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ as in (3.19c,3.19a, 3.19b) wherever it makes sense. We require that on the bifurcate sphere

$$\Re\frac{1}{\Xi'} > \frac{2}{M} \tag{4.22}$$

$$\mathcal{B}'^2 = 0 \tag{4.23}$$

$$(4\bar{\Xi}' - 2M)\nabla_X (P')^{-1} = 2\mathcal{F}(t, X) \tag{4.24}$$

for any $X \in T\mathfrak{H}_0$, and that the following are each satisfied at some (possibly different) point in $\mathfrak{H}_0$

$$t^2 + 1 = \Re(\frac{2M}{P'}) - \left|\frac{q_E + iq_B}{P'}\right|^2 \tag{4.25}$$

$$\Re(P') > M \tag{4.26}$$

Remark 4.4.1. *That the assumptions (AF) and (SBS) are reasonable have been, the author hopes, demonstrated in Chapter 1. See also Remark 1.1 in [14].*

The technical conditions (4.22) and (4.23) are those used in Corollary 4.2.5. Indeed, when $\Xi' \neq 0$, *(4.22) implies* $|\Xi'| \geq \Re\Xi' > 2|\Xi|^2/M$, *which implies* $4|\Xi'| < 2M$. *This condition should be compared with condition (1.7) in [14]. That this condition has to be prescribed on the entire bifurcate sphere and not just at a point is a complication introduced by the Maxwell structure. The condition (4.24) is a relaxation of the condition (1.6) in [14] (in essence the former is the latter's derivative). Observe* *that if we have a priori knowledge that the Maxwell field vanishes identically, (4.24) and (4.25) together implies condition (1.6) in [14]. In general, (4.24) and (4.25) are the minimal conditions required to recover the crucial Lemma 7.4 in [14]; see also Lemma 3.3.7 in Chapter 3.*

Remark 4.4.2. *The technical condition (4.25) is necessary in view of the local version of the isometry theorems (see Theorem 3.2.1 herein and Theorem 1 in [23]), as they essentially fix the one remaining free constant (C_4 in Theorem 3.2.1) to allow the derivation of Lemma 3.3.7. This constant can be interpreted as carrying information from spatial infinity; it is through the precise value of this constant that we can use the condition that our local neighborhood can be embedded in a space-time that is stationary asymptotically flat. One should compare with it the constants M and $q_E + iqB$, which can be made arbitrary (as long as $M > |q_E + iq_B|$) without impact on most of the proof.*

Remark 4.4.3. *The conditions (4.22) and (4.26) are relatively mild: they are manifestations of the requirement that the black hole is non-extremal: that $M^2 - a^2 - |q_E + iq_B|^2 > 0$.*

The alignment conditions (4.23), (4.24), and (4.25) are the main technical assumptions. They represent some sort of rigidity assumption on the bifurcate sphere of a black hole solution. It is hoped that they may be eventually removed.

Lemma 4.4.4. *Under the assumption (AF), (SBS), (T), the quantity $\Re(P')$ is constant on $\mathfrak{H}^{\pm}$, and by (4.25) and (4.26) we have $M < \Re(P') \leq 2M$ on $\mathfrak{H}_0$. As a consequence $\mathcal{Q}'_{abcd}$ is well-defined in a neighborhood of $\mathfrak{H}_0$.*

Proof. Let x_0 be the point on which (4.25) holds. Since t^a is space-like, this implies that $|P'| \leq 2M$ at the point by rearranging the algebraic identity. Therefore P' is well-defined in a small neighborhood $N \subset \mathfrak{H}_0$. Corollary 4.2.5 implies that $\mathcal{Q}'_{abcd}$, $\mathcal{B}'_{ab}$ and their first derivatives vanish on N. In view of the computations leading up to

Lemma 3.3.7, in particular (3.5), one sees that the same decomposition holds and therefore $\Re(P') = y$ is constant on N. Therefore P' is well-defined on the entirety of $\mathfrak{H}_0$. By (4.25) and (4.26), it is then clear that $M < \Re(P') \leq 2M$. □

We now state the main theorem of this chapter

Theorem 4.4.5 (Conditional Uniqueness of Kerr-Newman)**.** *Let $(\mathcal{M}, g_{ab}, \mathcal{H}_{ab})$ represent a smooth stationary space-time solving the Einstein-Maxwell equations. Let t denote the stationary Killing vector-field and assume that $\mathcal{H}_{ab}$ is also fixed by the symmetry generated by t. Also assume (AF), (SBS), and (T) as above. Then the exterior region $\mathfrak{D} \in \mathcal{M}$ is everywhere locally isometric to the Kerr-Newman solution.*

Here we'll give a quick overview of the method of proof. By the assumption that $\mathfrak{H}^{\pm}$ intersects transversely at the bifurcate sphere and the assumption that the black hole is non-extremal, we can apply the V-conditional Carleman estimate with V being the zero vector field, and the pseudo-convex weights given by the double-null foliation, in a small neighborhood of $\mathfrak{H}_0$, which shows that $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ both vanish in a neighborhood of $\mathfrak{H}_0$. The vanishing of $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ implies that we can use the Characterization Theorem 3.2.1, in particular Lemma 3.3.7. The conditions for applying the lemma are all satisfied in view of the technical assumptions (T). Therefore in the neighborhood we obtain control for $y = \Re(P')$.

To extend beyond the first neighborhood, we again apply the Carleman estimate now with V being the Killing vector field t and the pseudo-convex weights being the radial function y. By construction y is "increasing as we get away from the black hole". Since $\nabla y = \nabla \Re(\Xi')^{-1}$, we see that condition (4.22) will be satisfied uniformly, and we will have control on $(4\Xi' - 2M)^{-1}$ throughout. Therefore by looking at the form of the wave equation (4.15), we can continue the Carleman estimate as long as P' remains bounded. But by technical assumption (4.25), and the argument given in Section 3.4, P' cannot blow-up at any finite Riemannian distance from the bifurcate sphere (more precisely, using the Riemannian metric on Σ_0, y cannot blow-up at any finite distance from $\mathfrak{H}_0$), and so we can cover the entire exterior region.

The remainder of this section will be used to realize the above heuristics.

4.4.1 The first neighborhood

First we give some quantitative control on the double null foliation constructed in Section 2.5.2. Recall that O_ϵ is defined as a neighborhood of $\mathfrak{H}_0$ such that the optical functions $|u|, |\underline{u}|$ are bounded by ϵ. Fix ϵ_0 as in Section 2.5.2 such that $\Omega > \frac{1}{2}$ on O_{ϵ_0}. By assumption (AF), t intersects Σ_0 transversally, so $|g(t, T_0)| > 0$ on $\Sigma_0 \cap \mathfrak{D}$. Therefore we have that for any small $0 < \epsilon < \epsilon_0$, there is a corresponding large constant $\tilde{A}_\epsilon$ such that

$$|g(t, T_0)| > \frac{1}{\tilde{A}_\epsilon} \ , \qquad \forall x \in (\Sigma_0 \cap \mathfrak{D}) \setminus O_\epsilon \ . \tag{4.27}$$

With a possible reduction on ϵ_0, we can require that there exists a constant A_0 such that

$$\frac{u}{\underline{u}} + \frac{\underline{u}}{u} \leq A_0 \ , \qquad \forall x \in O_{\epsilon_0} \cap \Sigma_0 \cap \mathfrak{D} \ , \tag{4.28}$$

(this reflects the fact that Σ_0 is space-like and the level-surfaces of $u, \underline{u}$ are null). Similarly, in view of Lemma 4.4.4, we can require that ϵ_0 is chosen small enough such that P', $\mathcal{B}'_{ab}$, and $\mathcal{Q}'_{ab}$ is well-defined on O_{ϵ_0}.

We now construct a suitable set of coördinates in a tubular neighborhood of Σ_0 following [14]. By possibly enlarging the constant A_0 above, we can arrange so that at every point $x_0 \in \Sigma_0 \cap \bar{\mathfrak{D}}$, there exists a diffeomorphism Π^{x_0} from the ball of radius 1 B_1, centered at the origin, to a neighborhood $B(x_0) \subset \mathcal{M}$, with $\Pi^{x_0}(0) = x_0$, and satisfying

$$\sup_{x_0\in\Sigma_0\cap\bar{\mathfrak{D}}}\sup_{x\in B_1}\sum_{j=0}^{6}\sum_{\beta,\gamma=1}^{4}|\partial^{(j)}(\Pi_*^{x_0}g)_{\beta\gamma}|+|\partial^{(j)}(\Pi_*^{x_0}g)^{-1}_{\beta\gamma}|\leq A_0 \tag{4.29a}$$

$$\sup_{x_0\in\Sigma_0\cap\bar{\mathfrak{D}}}\sup_{x\in B_1}\sum_{j=0}^{6}\sum_{\beta=1}^{4}|\partial^{(j)}(\Pi_*^{x_0}t)^{\beta}|\leq A_0 \tag{4.29b}$$

$$\sup_{x_0\in\Sigma_0\cap\bar{\mathfrak{D}}}\sup_{x\in B_1}\sum_{j=0}^{5}\sum_{\alpha,\beta=1}^{4}|\partial^{(j)}(\Pi_*^{x_0}H)_{\alpha\beta}|\leq A_0 \tag{4.29c}$$

where $(\Pi_*^{x_0}g)_{\beta\gamma}$ is the matrix representing the pull-back of the metric g, similarly $(\Pi_*^{x_0}g)^{-1}_{\beta\gamma}$ is the inverse matrix, $(\Pi_*^{x_0}t)^{\beta}$ denotes the coördinate coefficients of the vector field on B_1 representing the Killing vector field t^a, and $(\Pi_*^{x_0}H)_{\alpha\beta}$ is the coördinate coefficients of the pull-back two-form corresponding to the Maxwell field. Such a choice of diffeomorphisms is always possible on any compact region; that we can do it for all of $\Sigma_0\cap\bar{\mathfrak{D}}$ is due to asymptotic flatness. Now let $\tilde{\mathcal{M}} := \cup_{\Sigma_0\cap\bar{\mathfrak{D}}}B(x_0)$. We can arrange for $\tilde{\mathcal{M}}$ to be simply connected.

The compactness of $\mathfrak{H}_0$ also allows us to assume that A_0 is chosen so that

$$\sup_{x_0\in\mathfrak{H}_0}\sup_{x\in B_1}\left[\sum_{j=0}^{6}\left(|\partial^{(j)}\Pi_*^{x_0}u|+|\partial^{(j)}\Pi_*^{x_0}\underline{u}|\right)+\sum_{\alpha=1}^{4}\left(|\frac{\partial}{\partial x^\alpha}\Pi_*^{x_0}u|^{-1}+|\frac{\partial}{\partial x^\alpha}\Pi_*^{x_0}\underline{u}|^{-1}\right)\right]\leq A_0 \tag{4.30}$$

By compactness of $\mathfrak{H}_0$ again, we can require that A_0 is chosen such that we have "room" in (4.22) and (4.26):

$$\Re\frac{1}{\Xi'}>\frac{2}{M}(1+A_0^{-1})\qquad \forall x_0\in\mathfrak{H}_0 \tag{4.31a}$$

$$\Re(P')>M(1+A_0^{-1})\qquad \text{for some } x_0\in\mathfrak{H}_0 \tag{4.31b}$$

Lastly, we also require $A_0\geq\epsilon_0^{-1}$. For the rest of the proof, ϵ_0 and A_0 will be fixed constants. We will also write C_{A_0} for an arbitrary constant that depends polynomially on A_0. Between different expressions C_{A_0} may be different.

Remark 4.4.6. *The control given by (4.29) implies that we have control of the coördinate expressions of the following in terms of a polynomial function of* A_0*:*

- F_{ab} *up to five derivatives,*
- *the first six (but not the zeroth order) derivatives of* Ξ'*,*
- *when* $|4\Xi' - 2M| > A_0^{-1}$*, and when* $\mathcal{F}^2 > A_0^{-1}$*, control of* P' *and its first five derivatives,*
- $\mathcal{B}'_{ab}$ *up to five derivatives,*
- *when* $|4\Xi' - 2M| > A_0^{-1}$*, and when* $\mathcal{F}^2 > A_0^{-1}$*, control of* $\mathcal{Q}'_{abcd}$ *up to five derivatives.*

Lastly, we observe that by Taylor's theorem with remainders, if X *is an object with which we have control of first* k *derivatives, and assume that* X *vanishes on the bifurcate horizon, then we can write* $X = u\underline{u}X'$ *for some smooth function* X' *whose first* $k-2$ *derivatives we can control.*

Now, let the weight function $h_\epsilon = \epsilon^{-1}(u+\epsilon)(\underline{u}+\epsilon)$ be defined in O_{ϵ^2} for $0 < \epsilon < \epsilon_0$. Also let $N^{x_0} : B(x_0) \to [0,1)$ be the function

$$N^{x_0}(x) = |(\Pi^{x_0})^{-1}(x)|^2 \tag{4.32}$$

where the norm is the Euclidean norm. The following Carleman estimate is a consequence of the bifurcate null geometry, and does not depend on the Einstein-Maxwell equations.

Lemma 4.4.7 (Ionescu-Klainerman, see Lemma 6.2 in [14])**.** *There is* $\epsilon \in (0, \epsilon_0)$ *sufficiently small and depending on* A_0*, and* $\tilde{C}_\epsilon$ *sufficiently large, such that for any* $x_0 \in \mathfrak{H}_0$*, any* $\lambda \geq \tilde{C}_\epsilon$*, and any* $\phi \in C_0^\infty(\Pi^{x_0} B_{\epsilon^{10}})$*,*

$$\lambda\|e^{-\lambda f_\epsilon}\phi\|_{L^2} + \|e^{-\lambda f_\epsilon}|\partial^{(1)}\phi|\|_{L^2} \leq \tilde{C}_\epsilon \lambda^{-1/2}\|e^{-\lambda f_\epsilon}\Box_g\phi\|_{L^2} \tag{4.33}$$

where $f_\epsilon = \ln(h_\epsilon + \epsilon^{12} N^{x_0})$*.*

The proof of the lemma, which we omit here, is an application of the general Carleman inequality Proposition 4.3.3 with V being the zero vector field. Using the above lemma, we obtain

Proposition 4.4.8. *There exists* $r_1 = r_1(A_0) > 0$ *such that* $\mathcal{Q}'_{abcd}$ *and* $\mathcal{B}'_{abcd}$ *vanishes in* $O_{r_1} \cap \mathfrak{D}$*.*

Proof. The proof is a direct adaptation of the proof for Proposition 6.1 in [14]. We include it here for completeness. Let $\epsilon = \epsilon(A_0)$ be fixed by the previous lemma. Write $\mathcal{S}$ as the vector valued function whose entries are $\mathcal{B}'(\partial_\alpha, \partial_\beta)$, $\nabla\mathcal{B}'(\partial_\alpha, \partial_\beta, \partial_\gamma)$, and $\mathcal{Q}'(\partial_\alpha, \partial_\beta, \partial_\gamma, \partial_\delta)$. By definition of ϵ_0, $\mathcal{S}$ is smooth and satisfies a wave equation with smooth coefficients. We therefore have the estimate

$$|\Box_g \mathcal{S}|_{\ell^\infty} \leq C_{A_0} |\partial^{(1)} \mathcal{S}|_{\ell^\infty} + |\mathcal{S}|_{\ell^\infty} \tag{4.34}$$

in $B_{\epsilon^{10}}$.

To apply the Carleman estimate, however, we need a function with compact support. So we apply a cut-off to show that $\mathcal{S}$ vanishes identically in $B_{\epsilon^{40}} \cap \mathfrak{D}$. Fix x_0 in $\mathfrak{H}_0$, and let $\eta : \mathbb{R} \to [0,1]$ supported in $[1/2, \infty)$ and equals to 1 on $[3/4, \infty)$. For arbitrary $\delta \in (0,1]$, define

$$\mathcal{S}^{\delta,\epsilon} = \mathcal{S} 1_{\mathfrak{D}} \eta(u\underline{u}/\delta)(1 - \eta(N^{x_0}/\epsilon^{20})) = \mathcal{S}\tilde{\eta}^{\delta,\epsilon} \ . \tag{4.35}$$

By construction it has compact support in $B_{\epsilon^{10}}$ so we can apply the Carleman estimate.

Compute

$$\Box_g \mathcal{S}^{\delta,\epsilon} = \tilde{\eta}^{\delta,\epsilon} \Box_g \mathcal{S} + 2\nabla^a \mathcal{S} \nabla_a \tilde{\eta}^{\delta,\epsilon} + \mathcal{S} \Box_g \tilde{\eta}^{\delta,\epsilon} \ .$$

By letting λ be sufficiently large in the Carleman estimate, we have that

$$\lambda \| e^{-\lambda f_\epsilon} \tilde{\eta}^{\delta,\epsilon} |\mathcal{S}|_{\ell^\infty} \|_{\mathcal{L}^2} \leq$$
$$\tilde{C}_\epsilon \left(\| e^{-\lambda f_\epsilon} |\nabla^a \mathcal{S} \nabla_a \tilde{\eta}^{\delta,\epsilon}|_{\ell^\infty} \|_{L^2} + \| e^{-\lambda f_\epsilon} |\mathcal{S}|_{\ell^\infty} (|\Box_g \tilde{\eta}^{\delta,\epsilon}| + |\partial^{(1)} \tilde{\eta}^{\delta,\epsilon}|) \|_{L^2} \right) \ .$$

Now observe that we can define two sets

$$G^\delta := \{ x \in B_{\epsilon^{10}} \cap \mathfrak{D} : u(x)\underline{u}(x) \in (\delta/2, \delta) \}$$

and

$$G^\epsilon := \{ x \in B_{\epsilon^{10}} \cap \mathfrak{D} : N^{x_0}(x) \in (\epsilon^{20}/2, \epsilon^{20}) \}$$

where, by construction, $|\Box_g \tilde{\eta}^{\delta,\epsilon}| + |\partial^{(1)} \tilde{\eta}^{\delta,\epsilon}|$ is supported. We claim now that

$$|\Box_g \tilde{\eta}^{\delta,\epsilon}| + |\partial^{(1)} \tilde{\eta}^{\delta,\epsilon}| \leq C_\eta (\delta^{-1} 1_{G^\delta} + 1_{G^\epsilon}) \ . \tag{4.36}$$

For the term $|\partial^{(1)} \tilde{\eta}^{\delta,\epsilon}|$ the estimate follows from the definition. For the term with the D'Alembertian, we consider the definition

$$|\Box_g \tilde{\eta}^{\delta,\epsilon}| \leq |\Box_g (1_{\mathfrak{D}} \eta(u\underline{u}/\delta))| (1 - \eta(N^{x_0}/\epsilon^{20})) + C_\eta (\delta^{-1} 1_{G^\delta} + 1_{G^\epsilon})$$

The only term in $\Box_g \eta(u\underline{u}/\delta)$ that can give problem is when by chain rule we obtain $\eta'' \delta^{-2} (\nabla^a (u\underline{u}) \nabla_a (u\underline{u}))$. But using the eikonal equations for u and $\underline{u}$, we see that

$$|\nabla^a (u\underline{u}) \nabla_a (u\underline{u})| = 2|\Omega||u\underline{u}|$$

and using that η'' only is supported when $u\underline{u} \in (\delta/2, \delta)$, we see that

$$|\eta'' \delta^{-2} (\nabla^a (u\underline{u}) \nabla_a (u\underline{u}))| < C_\eta \delta^{-1} 1_{G^\delta}$$

as desired.

For the term $\nabla^a \mathcal{S} \nabla_a \tilde{\eta}^{\delta,\epsilon}$, we use the fact that since $\mathcal{S}$ is smoothly defined and vanishes on $\mathfrak{H}^\pm$, we can write $\mathcal{S} = u\underline{u}\mathcal{S}'$ for some smooth $\mathcal{S}'$. By the same argument as was just given,

$$|\nabla^a \mathcal{S} \nabla_a \tilde{\eta}^{\delta,\epsilon}| \leq C_{A_0} C_\eta (1_{G^\epsilon} + 1_{G^\delta}) \ .$$

Combining the computations above,

$$\lambda \| e^{-\lambda f_\epsilon} \tilde{\eta}^{\delta,\epsilon} |\mathcal{S}|_{\ell^\infty} \|_{\mathcal{L}^2} \leq \tilde{C}_\epsilon C_\eta C_{A_0} \| e^{-\lambda f_\epsilon} (1_{G^\delta} + 1_{G^\epsilon}) \|_{L^2} \ .$$

Now we take the limit as $\delta \to 0$, and observe that $\|1_{G^\delta}\|_{L^2} \to 0$, the inequality becomes

$$\lambda \| e^{-\lambda f_\epsilon} 1_{B_{\epsilon^{40}} \cap \mathfrak{D}} |\mathcal{S}|_{\ell^\infty} \|_{L^2} \leq \tilde{C}_\epsilon C_\eta C_{\mathcal{S}'} \| e^{-\lambda f_\epsilon} 1_{G^\epsilon} \|_{L^2} \ .$$

Now observe that by definition of the weight f_ϵ

$$\inf_{B_{\epsilon^{40}} \cap \mathfrak{D}} e^{-\lambda f_\epsilon} \geq \sup_{G^\epsilon} e^{-\lambda f_\epsilon}$$

so

$$\lambda \| 1_{B_{\epsilon^{40}} \cap \mathfrak{D}} |\mathcal{S}|_{\ell^\infty} \|_{L^2} \leq \tilde{C}_\epsilon C_\eta C_{\mathcal{S}'} \| 1_{G^\epsilon} \|_{L^2}$$

the right hand side now a fixed constant. Taking $\lambda \to \infty$ gives us that $\mathcal{S}$ must vanish in $B_{\epsilon^{40}} \cap \mathfrak{D}$. □

4.4.2 Consequences of vanishing $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{ab}$

Let $\mathcal{N} \subset \tilde{\mathcal{M}} \cap \bar{\mathfrak{D}}$ be a connected open set containing $\mathfrak{H}_0$ such that $P', \mathcal{B}'_{ab}, \mathcal{Q}'_{abcd}$ are smoothly defined (in particular $4\bar{\Xi}' \neq 2M$ and $\mathcal{F}^2 \neq 0$). Assume that $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ both vanish on $\mathcal{N}$, and the technical conditions (4.22), (4.25), (4.26) are satisfied. We observe that in the language of Theorem 3.2.1, the constants C_1, C_2 and C_4 are now fixed (the last because of (4.25)). Therefore we can appeal to Lemma 3.3.7 (or analogously Lemma 7.1 in [14] in the case of vanishing charge, since $\mathcal{B}'_{ab} = 0$ implies $\mathcal{H}_{ab} = 0$ in $\mathcal{N}$[2]) to get a constant $\mathfrak{A}$ such that, decomposing $P' = y + iz$,

$$(\nabla z)^2 = \frac{\mathfrak{A} - z^2}{y^2 + z^2} \;, \qquad (\nabla y)^2 = \frac{\mathfrak{A} + y^2 + |q_E + iq_B|^2 - 2My}{y^2 + z^2} \;.$$

Proposition 4.4.9. *In $\mathcal{N}$, $|\partial^{(1)} y|$ is uniformly bounded by C_{A_0}.*

Proof. Consider the set $\mathcal{N} \cap \{|P'| \leq 8M\}$. Using that $|q_E + iq_B|^2 \nabla P' = -2P'^2 \nabla \Xi'$ (a consequence of the vanishing of $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$), we see that in the non-vanishing charge case if $|P'| \leq 8M$, we can control $|\partial^{(1)} P'|$ trivially. In the case with vanishing charge, observe that $\nabla P' = \frac{2P'^2}{M} \mathcal{F}(t, \cdot)$, so again in $|P'| \leq 8M$ we have direct control. Therefore it suffices to consider $\mathcal{N} \cap \{|P'| > 8M\}$. But if $|P'| > 8M$, $\Re(2M/P') < 1/4$. So by (4.25) (which by the arguments in Chapter 3 is extended to an algebraic identity on $\mathcal{N}$) $t^2 < -3/4$. In other words, t is time-like. Since $\nabla_t y = 0$ by definition, ∇y is a space-like vector on $\mathcal{N} \cap \{|P'| > 8M\}$. The uniform bound $t^2 < -3/4$ implies that we can uniformly control

$$|\partial^{(1)} y|^2 < C_{A_0} (\nabla y)^2 = C_{A_0} \frac{y^2 - 2My + \mathfrak{A} + |q_E + iq_B|^2}{y^2 + z^2}$$

Since z is a bounded function by Lemma 3.3.7, the right hand side is bounded by a large constant multiple of C_{A_0} if $|P'| > 8M$. □

Now let $\mathcal{N}' \subset \mathcal{N} \cap \mathfrak{D}$ be defined such that additionally,

$$y^2 - 2My + \mathfrak{A} + |q_E + iq_B|^2 > 0 \;.$$

We will compute the connection coefficients and the Hessian of y using the tetrad

[2] It is not necessary to appeal to Lemma 7.1. In view of the definitions given in Theorem 3.5.1, the computations in Chapter 3 can be carried through exactly if $\mathcal{F}^2$ is assumed to be non-vanishing.

formalism. The condition above, by (3.17), implies that $U = l_a t^a \underline{l}_b t^b = \frac{1}{2}(\nabla y)^2 \neq 0$, where $l, \underline{l}$ are principal null vectors defined in Chapter 3. So we will impose the normalization condition that $g(t, \underline{l}) = 1$ for the local null tetrad.

As we have already computed in the proof of the characterization theorem, we decompose $\nabla_a y = -l_a + U\underline{l}_a$, and hence that in an adapted tetrad, we have the following conditions

$$\begin{aligned} \underline{\theta} &= 1/\bar{P}' & \theta &= -U/P' \\ \xi = \underline{\xi} &= 0 & \underline{\vartheta} = \vartheta &= 0 \\ \zeta &= \eta & \eta\bar{\eta} &= (\nabla z)^2/(2P'\bar{P}') \end{aligned}$$

(the equations in the third line follows from (2.22f) and Lemma 3.3.7). Furthermore, we have the following additional conditions: by (2.22b), $D\theta = -\theta^2 - \omega\theta$, so we can solve for

$$\omega = -DU/U = \frac{M(y^2 - z^2) - (\mathfrak{A} + |q_E + iq_B|^2 - z^2)y}{(y^2 + z^2)^2} \tag{4.37}$$

and that

$$\underline{\omega} = 0 \; . \tag{4.38}$$

Lastly, we wish to compute the Hessian of y. To do so we use the formula $(\nabla^2 y)_{\alpha\beta} = e_\alpha(e_\beta y) - \Gamma^\mu{}_{\beta\alpha} e_\mu(y)$ and read off the values. In conclusion, the computations lead us to the following

Lemma 4.4.10. *On the set $\mathcal{N}'$ where $(\nabla y)^2 > 0$, choosing the null tetrad to normalize $g(t, \underline{l}) = 1$, we define the functions*

$$U = \frac{y^2 - 2My + \mathfrak{A} + |q_E + iq_B|^2}{2(y^2 + z^2)} \; ,$$

and

$$H = \frac{M(y^2 - z^2) - (\mathfrak{A} + |q_E + iq_B|^2 - z^2)y}{(y^2 + z^2)(y^2 - 2My + \mathfrak{A} + |q_E|iq_B|^2)} \ .$$

Then we have the following identities:

$$\delta y = \bar{\delta} y = Dz = \underline{D}z = 0 \ , \qquad \underline{D}y = 1 \ , Dy = -U \ , (\nabla y)^2 = 2U \tag{4.39}$$

$$(\nabla z)^2 = \frac{\mathfrak{A} - z^2}{y^2 + z^2} \ , z^2 \leq \mathfrak{A} \tag{4.40}$$

$$t^a = -l^a - U\underline{l}^a - \bar{\eta} P' m^a - \eta \bar{P}' \bar{m}^a \tag{4.41}$$

and

$$\underline{\theta} = 1/\bar{P}' \qquad \theta = -U/P' \tag{4.42}$$

$$\xi = \underline{\xi} = 0 \qquad \underline{\vartheta} = \vartheta = 0 \tag{4.43}$$

$$-\underline{\eta}\frac{P'}{\bar{P}'} = \zeta = \eta \qquad \eta\bar{\eta} = \frac{(\nabla z)^2}{2(y^2 + z^2)} \tag{4.44}$$

$$\omega = HU \qquad \underline{\omega} = 0 \tag{4.45}$$

We also have the following expressions for the Hessian of y:

$$(\nabla^2 y)_{33} = (\nabla^2 y)_{44} = 0 \qquad (\nabla^2 y)_{34} = (\nabla^2 y)_{43} = -HU \tag{4.46}$$

$$(\nabla^2 y)_{41} = (\nabla^2 y)_{14} = \underline{\eta} U \qquad (\nabla^2 y)_{31} = (\nabla^2 y)_{13} = -\zeta \tag{4.47}$$

$$(\nabla^2 y)_{42} = (\nabla^2 y)_{24} = \bar{\underline{\eta}} U \qquad (\nabla^2 y)_{32} = (\nabla^2 y)_{23} = -\bar{\zeta} \tag{4.48}$$

$$(\nabla^2 y)_{12} = (\nabla^2 y)_{21} = \frac{2y}{y^2 + z^2} U \qquad (\nabla^2 y)_{22} = (\nabla^2 y)_{11} = 0 \tag{4.49}$$

Remark 4.4.11. *Observe that the above lemma is formally identical to Lemma 7.3 in [14].*

The vanishing of $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ also gives us finer control on y on a subset of O_{r_1}:

Lemma 4.4.12. *There is a constant $y_{\mathfrak{H}} \in (M, 2M]$ such that $y = y_{\mathfrak{H}}$ on the horizon. In addition, $\mathfrak{A} \in [0, M^2 - |q_E + iq_B|^2)$, and that for sufficiently small $\epsilon = \epsilon(A_0)$ we have that*

$$y > y_{\mathfrak{H}} + MC_{A_0}^{-1} u\underline{u}$$

on $O_\epsilon \cap \mathfrak{D}$.

Proof. By Lemma (4.4.4) we've already shown the first claim. Using that $(\nabla P)^2 = 0$ on $\mathfrak{H}_0$, we see that $(\nabla y)^2 = 0$ and hence $y_{\mathfrak{H}}^2 - 2My_{\mathfrak{H}} + \mathfrak{A} + |q_E + iq_B|^2 = 0$. Since $y_{\mathfrak{H}}^2 > M$, we must have that $\mathfrak{A} + |q_E + iq_B|^2 < M^2$.

Now notice that $y - y_{\mathfrak{H}}$ is a smooth function in $O_{r_1} \cap \mathfrak{D}$ that vanishes on $\mathfrak{H}^\pm$. Therefore we can write it as $y - y_{\mathfrak{H}} = u\underline{u}y'$ for some smooth function y' whose first derivatives are bounded by C_{A_0}. It thus suffices to show that y' does not vanish on the horizon. Using the identity

$$\Box_g P' = -\frac{2}{P'\bar{P}}(M - \bar{P}')$$

we get

$$\Box_g y = -\frac{2}{y^2 + z^2}(M - y)\ .$$

Applying this to $y = y_{\mathfrak{H}} + u\underline{u}y'$ and evaluating on the horizon:

$$\frac{2}{y_{\mathfrak{H}}^2 + z^2}(y_{\mathfrak{H}} - M) = \Box_g(u\underline{u}y')|_{\mathfrak{H}^\pm} = 2\nabla^a u \nabla_a \underline{u} y' = 2y'\ .$$

Since we assumed that $y_{\mathfrak{H}} > M(1 + A_0^{-1})$, $y' > MC_{A_0}^{-1}$ on the horizon, and therefore for some sufficiently small ϵ, we have that $y' > MC_{A_0}^{-1}$ in $O_\epsilon \cap \mathfrak{D}$ as desired. □

4.4.3 The bootstrapping

In view of the assumption (AF), we just need to show that $\mathcal{S}$ vanishes along $\Sigma_0 \cap \mathfrak{D}$. We will here use a bootstrap argument to show that $4\Xi' - 2M \neq 0$, $\frac{1}{P'} \neq 0$, $\mathcal{B}'_{ab} = 0$

and $\mathcal{Q}'_{abcd} = 0$ on the stated set. Then by Theorem 3.5.1 we can conclude Theorem 4.4.5.

We define some sets over which our induction will work. Let Σ_0' be the open subset of $\Sigma_0 \cap \mathfrak{D}$ where $\mathcal{F}^2 \neq 0$ and $4\Xi' \neq 2M$. Clearly Σ_0' contains $O_{\epsilon_0} \cap \mathfrak{D}$ by definition. For any $R > y_{\mathfrak{H}}$ define $\mathcal{V}_R = \{x \in \Sigma_0' : y < MR\}$, and $\mathcal{U}_R$ as the unique connected component of $\mathcal{V}_R$ whose closure in Σ_0 contains $\mathfrak{H}_0$.

Proposition 4.4.13. *There exists a number $R_1 \geq y_{\mathfrak{H}}/M + C_{A_0}^{-1}$ such that $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ vanish in $\mathcal{U}_{R_1}$.*

Proof. Let O_ϵ be as in Lemma 4.4.12. $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ clearly vanishes in it. By construction, $u/\underline{u} + \underline{u}/u \leq A_0$ in $\Sigma_0 \cap \mathfrak{D} \cap O_\epsilon$. So by Lemma 4.4.12,

$$MC_{A_0}^{-1}(u^2 + \underline{u}^2) \leq y - y_{\mathfrak{H}} \leq MC_{A_0}(u^2 + \underline{u}^2)$$

on $\Sigma_0 \cap \mathfrak{D} \cap O_\epsilon$. Thus for sufficiently small R_1, $\mathcal{U}_{R_1} \subset O_\epsilon$. $\square$

The main result of this section will be

Proposition 4.4.14. *For any $R_2 \geq R_1$ defined above, the tensors $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ vanish identically in $\mathcal{U}_{R_2}$.*

We prove the proposition by induction, in view of Proposition 4.4.13. Therefore it suffices to show that given any $R_2 \geq R_1$, assuming that $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ vanish identically in $\mathcal{U}_{R_2}$, then there exists a small r that depends only on C_{A_0}, $\tilde{A}_\epsilon$ (see (4.27). This constant controls the fact that t intersects Σ_0 transversally in $\mathfrak{D}$. We choose a small enough ϵ so that $O_\epsilon \cap \mathfrak{D} \cap \Sigma_0 \subset \mathcal{U}_{R_1}$ from above), and the radius R_2, such that $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ also vanish in $\mathcal{U}_{R_2+r}$. In the following C_{R_2} will denote any constants depending on R_2, A_0, and $\tilde{A}_\epsilon$. To close the induction, it is crucial that r only depends on the above listed constants.

Using the computations in Chapter 3, one recalls that in a domain where $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ vanish, $|q_E + iq_B|^2/P' = 2\Xi'$. Of course, in the case where the charge vanishes, Ξ' is identically 0, so $(4\Xi' - 2M)$ is a constant and can be safely ignored in our arguments. In the case where the charge does not vanish, we see that $2M - 4\Xi' = 2[-|q_E+iq_B|^2(y-iz)+M(y^2+z^2)]/(y^2+z^2)$. Recall that $z^2 \leq \mathfrak{A} < M^2 - |q_E+iq_B|^2$. Consider a point on the boundary $\partial\mathcal{U}_{R_2}$[3], $y = MR_2$ there, so we have the very loose bound

$$\frac{2(R_2^2+1)}{R_2^2}M > |2M - 4\Xi'| > \frac{2(R_2^2 - R_2)}{(R_2^2+1)}M$$

on $\partial\mathcal{U}_{R_2}$. Using that $R_1 \geq 1$ by construction,

$$4M > |2M - 4\Xi'| > \frac{R_2 - 1}{R_2}M \ .$$

Given the uniform control on the derivative of Ξ', there exists a small r_2' that only depends on A_0 and R_2 such that $|2M - 4\Xi'| \in (\frac{R_2-1}{2R_2}M, 8M)$ inside $B_{r_2'}$ of a point on the boundary of $\mathcal{U}_{R_2}$. Now, since $\mathcal{F}^2 = -(P')^{-4}(4\bar{\Xi}' - 2M)^2$, on the boundary of $\mathcal{U}_{R_2}$

$$\frac{4}{M^3R_2^4} > |\mathcal{F}^2| > \frac{R_2-1}{4R_2^5}M^{-3}$$

so we can also choose r_2' in a manner only depending on A_0 and R_2 so that $|\mathcal{F}^2|$ is bounded, and bounded away from zero in $B_{r_2'}$ using the fact that $\mathcal{F}^2$ is smooth. Therefore P' is well-defined in $B_{r_2'}$ and so is y. We also have that the first four derivatives of y are controlled by C_{R_2}. So choosing r_2' sufficiently small again, we have

$$y \in ((y_{\mathfrak{H}} + MR_1)/2, 2MR_2) \ , \quad \forall x \in B_{r_2'} \ .$$

We consider r_2' fixed relative to a given R_2 from now on.

By (4.27), we can also fix a $\delta_2 > C_{R_2}^{-1}$ so that $(-\delta_2, \delta_2) \times (B_{r_2'} \cap \Sigma_0)$ is diffeomor-

[3]Since $\mathcal{U}_{R_2}$ is only defined as a subset of Σ_0, we consider its boundary only in $\Sigma_0 \cap \mathfrak{D}$. This will be taken as a definition hereon.

phic to $\cup_{|s|<\delta_2}\Phi_s^{(t)}(B_{r_2'}\cap\Sigma_0)$ where $\Phi_s^{(t)}$ is the one-parameter (s) family of isometries corresponding to the Killing vector field t. We write π^Q for the projection from $\cup_{|s|<\delta_2}\Phi_s^{(t)}(B_{r_2'}\cap\Sigma_0)$ to $B_{r_2'}\cap\Sigma_0$ induced by the above diffeomorphism.

We now write $\mathcal{N}_{R_2}$ for the connected component of $[\cup_s\Phi_s^{(t)}(\mathcal{U}_{R_2})\cup O_{r_1}]\cap\tilde{\mathcal{M}}$ that contains $\mathcal{U}_{R_2}$. By construction, we see that since t is Killing and the symmetry descends to all geometric quantities concerned, $4\Xi'\neq 2M$, $\mathcal{F}^2\neq 0$, and $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ are well-defined and vanishing in $\mathcal{N}_{R_2}$. Therefore our results from Section 4.4.2 and from Chapter 3 can be applied.

Lemma 4.4.15. *Consider a point x_0 on the boundary $\partial\mathcal{U}_{R_2}$. There exists some $r_2<r_2'$ such that*

$$\{\Pi^{x_0}(x), x\in B_{r_2}: y(x)<MR_2\}\subset\cup_{|s|<\delta_2}\Phi_s^{(t)}\mathcal{U}_{R_2}\ .$$

Proof. Recall that $(\nabla y)^2=(y^2-2My+\mathfrak{A}+|q_E+iq_B|^2)/(y^2+z^2)>0$ if $y>y_{\mathfrak{H}}$. Therefore we can find $r_2''<C_{R_2}^{-1}$ such that $(\nabla y)^2>C_{R_2}^{-1}$ in $B_{r_2''}$. Therefore there exists some $r_2<r_2''$ and a set B' such that $B_{r_2}\subset B'\subset B_{r_2''}$ and such that $\{x\in B': y(x)<M\mathcal{R}_2\}$ is connected. Thus $\pi^Q(\{x\in B': y(x)<M\mathcal{R}_2\})\subset B_{r_2'}\cap\Sigma_0$ is a connected set containing $\{x\in B'\cap\Sigma_0 : y(x)<M\mathcal{R}_2\}$. By the diffeomorphism above, y commutes with π^Q (which is a realization of the Killing symmetry). So $\pi^Q(\{x\in B': y(x)<M\mathcal{R}_2\})\subset\mathcal{U}_{R_2}$. The claim follows. □

We now localize our attention to $\mathcal{N}':=\mathcal{N}_{R_2}\cap\Pi^{x_0}B_{r_2}$. In $\mathcal{N}'$ we have that $(\nabla y)^2=2U>0$. So computations in the second half of Section 4.4.2 can be used. Note that $U\geq C_{R_2}^{-1}$ by construction, and that we can also estimate $H\geq C_{R_2}^{-1}$ by observing that $y-y_{\mathfrak{H}}\geq MC_{R_2}^{-1}$ and $y_{\mathfrak{H}}>M$.

We now offer a second Carleman estimate

Lemma 4.4.16. *There is an $\epsilon<r_2$ sufficiently small and $\tilde{C}_\epsilon$ sufficiently large such*

that for all $\lambda > \tilde{C}_\epsilon$ *and any* $\phi \in C_0^\infty(B_{\epsilon^{10}})$

$$\lambda\|e^{-\lambda f_\epsilon}\phi\|_{L^2} + \|e^{-\lambda f_\epsilon}|\partial^{(1)}\phi|\|_{L^2} \leq \tilde{C}_\epsilon \lambda^{-1/2}\|e^{-\lambda f_\epsilon}\Box_g\phi\|_{L^2} + \epsilon^{-6}\|e^{-\lambda f_\epsilon}\nabla_t(\phi)\|_{L^2} \quad (4.50)$$

where with $y(x_0) = R_2$,

$$f_\epsilon = \ln(y - R_2 + \epsilon + \epsilon^{12}N^{x_0}) \ . \quad (4.51)$$

Proof. We apply Proposition 4.3.3 with $V = t$, $h_\epsilon = y - R_2 + \epsilon$ and $e_\epsilon = \epsilon^{12}N^{x_0}$. It is clear that all the conditions are satisfied for ϵ sufficiently small except that we need to check h_ϵ is a good t-conditional family of pseudo-convex weights for some ϵ_1. By our construction, it is clear that (4.18a) it satisfied by definition. For (4.18b), we use the computations from Lemma 4.4.10

$$\nabla^a h_\epsilon \nabla_a h_\epsilon = (\nabla y)^2 = 2U$$
$$\nabla^a y = -l^a + U\underline{l}^a$$
$$\nabla^a y \nabla^b y \nabla^2_{ab} y = -2U(\nabla^2 y)_{34} = 2HU^2$$
$$\nabla^a h_\epsilon \nabla^b h_\epsilon (\nabla_a h_\epsilon \nabla_b h_\epsilon - \epsilon\nabla^2_{ab} h_\epsilon) = 4U^2 - 2\epsilon HU^2$$

which we can bound from below by ϵ_1^2 for sufficiently small ϵ_1.

It remains to check (4.18c). Since this is a point-wise condition, we decompose X using the null tetrad:

$$X = Wm + \bar{W}\bar{m} + Yl + Z\underline{l}$$

where W is complex and Y, Z are real. On the right hand side,

$$\begin{aligned}
g(X,X) &= 2W\bar{W} - 2YZ \\
g(t,X) &= Z + YU - W\eta\bar{P}' - \bar{W}\bar{\eta}P' \\
&= Z - YU + 2YU - 2\Re(W\zeta\bar{P}') \\
g(X,\nabla y) &= Z - YU \\
\nabla^2_{X,X}h_\epsilon &= 2(-YZHU + WY\underline{\eta}U - \zeta WZ + \bar{W}Y\underline{\bar{\eta}}U - \bar{W}\bar{\zeta}Z + \frac{2y}{y^2+z^2}UW\bar{W}) \\
&= \frac{4MR_2}{M^2R_2^2+z^2}UW\bar{W} - 2YZHU - 2\zeta W(Z + YU\frac{P'}{\bar{P}'}) - 2\bar{\zeta}\bar{W}(Z + YU\frac{\bar{P}'}{\bar{P}})
\end{aligned}$$

Notice that $\frac{P'}{\bar{P}'} = P'\frac{2MR_2}{M^2R_2^2+z^2} - 1$.

$$\nabla^2_{X,X}h_\epsilon = \frac{4MR_2U}{M^2R_2^2+z^2}|W|^2 - 2YZHU - (Z-YU)4\Re(\zeta W) - \frac{8MR_2}{M^2R_2^2+z^2}YU\Re(\zeta W\bar{P}')$$

So

$$\begin{aligned}
&\mu g(X,X) - \nabla^2_{XX}h_\epsilon + \epsilon^{-2}(|g(t,X)|^2 + |\nabla_X h_\epsilon|^2) \\
&= 2\mu|W|^2 - 2\mu YZ - \frac{4MR_2}{M^2R_2^2+z^2}UW\bar{W} + 2YZHU + 4(Z-YU)\Re(\zeta W) \\
&\quad + \frac{8MR_2}{M^2R_2^2+z^2}YU\Re(\zeta W\bar{P}') + 2\epsilon^{-2}(Z-YU)^2 \\
&\quad + 4\epsilon^{-2}(Z-YU)(YU - \Re(W\zeta\bar{P}')) + 4\epsilon^{-2}(YU - \Re(W\zeta\bar{P}'))^2 \\
&\geq (2\mu - \frac{4MR_2}{M^2R_2^2+z^2}U)|W|^2 + 2(Z-YU)(YHU - \mu Y + 2\Re(W\zeta\bar{P}')) \\
&\quad + 2Y^2(HU^2 - \mu U + \frac{4MR_2}{M^2R_2^2+z^2}U^2) \\
&\quad + \frac{8MR_2}{M^2R_2^2+z^2}YU(\Re(\zeta W\bar{P}') - YU) + 2\epsilon^{-2}(Z-YU)^2 \\
&\quad + 4\epsilon^{-2}(Z-YU)(YU - \Re(W\zeta\bar{P}')) + 4\epsilon^{-2}(YU - \Re(W\zeta\bar{P}'))^2
\end{aligned}$$

Now set $\mu = 3MR_2U/(M^2R_2^2 + z^2)$ and noting that $H > 0$,

$$\begin{aligned}\mu g(X,X) - \nabla^2_{XX}h_\epsilon + \epsilon^{-2}(|g(t,X)|^2 + |\nabla_X h_\epsilon|^2) \\ \geq \frac{2MR_2}{M^2R_2^2+z^2}U|W|^2 + \frac{2MR_2}{M^2R_2^2+z^2}U^2Y^2 \\ + \frac{\epsilon^{-2}}{4}(Z-YU)^2 + \frac{\epsilon^{-1}}{4}(YU - \Re(W\zeta\bar{P}')^2\end{aligned}$$

for sufficiently small ϵ. Noting that U is bounded from below by $C_{R_2}^{-1}$, this gives use

$$\begin{aligned}&\mu g(X,X) - \nabla^2_{XX}h_\epsilon + \epsilon^{-2}(|g(t,X)|^2 + |\nabla_X h_\epsilon|^2) \\ &\geq C_{R_2}^{-1}(Z^2 + Y^2 + |W|^2)\end{aligned}$$

and thus (4.18c) holds for ϵ_1 small enough. □

Using the Carleman estimate, we have that

Proposition 4.4.17. *For any fixed x_0 on $\partial\mathcal{U}_{R_2}$, there exists $r_3 < r_2$, r_3 depending on C_{R_2}, such that $\mathcal{B}'_{ab}$, and $\mathcal{Q}'_{abcd}$ vanishes in B_{r_3}.*

Proof. We apply Lemma 4.4.16 for the vector $\mathcal{S}$ defined as in the proof of Proposition 4.4.8, and we also defined the weight function η the same way. Write

$$\mathcal{S}^\epsilon = \mathcal{S}(1 - \eta(N^{x_0}/\epsilon^{40})) = \mathcal{S}\tilde{\eta}_\epsilon$$

and again commute the Carleman estimate with $\mathcal{S}^\epsilon$. The equation satisfied by $\mathcal{S}^\epsilon$ is

$$\begin{aligned}&\Box_g\mathcal{S}^\epsilon = \mathcal{A}\tilde{\eta}_\epsilon(\nabla\mathcal{S} + \mathcal{S}) + \nabla^a\mathcal{S}\nabla_a\tilde{\eta}_\epsilon + \mathcal{S}\Box_g\tilde{\eta}_\epsilon \\ &t(\mathcal{S}^\epsilon) = t(\tilde{\eta}_\epsilon)\mathcal{S} + t(\mathcal{S})\tilde{\eta}_\epsilon\end{aligned}$$

Using the wave equation satisfied by $\mathcal{S}$ and the fact that $\mathcal{B}'_{ab}$, $\nabla_c\mathcal{B}'_{ab}$, and $\mathcal{Q}'_{abcd}$ are stationary under t, we have the following differential inequalities

$$|\Box_g \mathcal{S}|_{\ell^\infty} \leq C_{R_2}(|\mathcal{S}|_{\ell^\infty} + |\partial^{(1)}\mathcal{S}|_{\ell^\infty})$$
$$|t(\mathcal{S})|_{\ell^\infty} \leq C_{R_2}|\mathcal{S}|_{\ell^\infty}$$

Putting this into the Carleman estimate, it is clear that for λ sufficiently large, it suffices to deal with the error terms coming from $\tilde{\eta}_\epsilon$. By construction, however, the support of the error terms lies in $\{y > MR_2\} \cap \{N^{x_0} \geq \epsilon^{50}\}$, the first contribution arising from the fact that $\mathcal{S}$ vanishes when $y \leq MR_2$ and the second from the fact that we are looking at the derivatives of a function which is constant near zero. Hence the error terms have the pointwise bound

$$|\nabla^a \mathcal{S} \nabla_a \tilde{\eta}_\epsilon| + |\mathcal{S}\Box_g \tilde{\eta}_\epsilon| + |\mathcal{S}||\partial^{(1)}\tilde{\eta}_\epsilon| \leq C_{R_2} 1_{\{y > MR_2\} \cap \{N^{x_0} \geq \epsilon^{50}\} \cap B_{\epsilon^{10}}}$$

Therefore arguing as in the proof of Proposition 4.4.8, we compare

$$\inf_{B_{\epsilon^{100}}} e^{-\lambda f_\epsilon} \geq \sup_{\{y > MR_2\} \cap \{N^{x_0} \geq \epsilon^{50}\} \cap B_{\epsilon^{10}}} e^{-\lambda f_\epsilon}$$

and the Carleman inequality implies

$$\lambda \left| \| 1_{B_{\epsilon^{100}}} \mathcal{S} \|_{L^2} \right|_{\ell^\infty} \leq C_{R_2} \tilde{C}_\epsilon \| 1_{\{y > MR_2\} \cap \{N^{x_0} \geq \epsilon^{50}\} \cap B_{\epsilon^{10}}} \|_{L^2}$$

and by taking $\lambda \to \infty$ we obtain that $\mathcal{S}$ must vanish identically in $B_{\epsilon^{100}}$. □

To finish the proof of Proposition 4.4.14, we need to show that for some $r < r_3$, $\mathcal{U}_{R_2+r}$ is contained in the set where we have shown $\mathcal{S}$ and hence $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ vanish. Consider the set

$$\mathcal{U}^r_{R_2} := \mathcal{U}_{R_2} \cup \left(\cup_{\partial \mathcal{U}_{R_2}} \{x \in B_{r_3/C} \cap \Sigma_0 : y(x) < M(R_2 + r)\} \right)$$

where the constant C is chosen so that the closure

$$\overline{\bigcup_{\partial\mathcal{U}_{R_2}} \{x \in B_{\frac{r_3}{C}} \cap \Sigma_0 : y(x) < M(R_2+r)\}} \subset \bigcup_{\partial\mathcal{U}_{R_2}} \overline{\{x \in B_{\frac{r_3}{4}} \cap \Sigma_0 : y(x) < M(R_2+r)\}}$$

where the closure is taken in Σ_0. C exists as $\partial\mathcal{U}_{R_2}$ is compact for any given R_2 by the asymptotic flatness assumption.

We claim that $\mathcal{U}_{R_2+r} \subset \mathcal{U}^r_{R_2}$. The proof is exactly the same of that in Section 8.2 of [14]. We'll sketch the proof here. Suppose the claim is false at some point p. Since $\mathcal{U}_{R_2+r}$ is, by definition, connected to $\mathfrak{H}_0$, we can choose a path in $\mathcal{U}_{R_2+r}$ such that it contains points not in $\mathcal{U}^r_{R_2}$. Take one such smooth path, parametrize it to start from $\mathfrak{H}_0$, and let p' be the first point not in $\mathcal{U}^r_{R_2}$. So p' is necessarily in $\overline{\mathcal{U}^r_{R_2}}$. Now p' cannot lie in $\overline{\mathcal{U}}_{R_2}$, since by definition all those points are interior points of $\mathcal{U}^r_{R_2}$. So

$$p' \in \cup_{\partial\mathcal{U}_{R_2}} \{x \in B_{r_3/C} \cap \Sigma_0 : y(x) < M(R_2+r)\} \ .$$

But we defined the closure in such a way that p' must lie in $B_{r_3/2} \cap \Sigma_0$ for some boundary point $x_0 \in \partial\mathcal{U}_{R_2}$. Inside $B_{r_3/2}$, we have that ∇y is a space-like smooth vector field, and we can flow p' along it in the negative direction. If r is chosen small enough (say $r < r_3/1000$), this operation generates (via a projection onto Σ_0 by π^Q) a smooth curve the lies entirely in $B_{r_3/2} \cap \Sigma_0$ connecting p' to some point p'' in $B_{r_3/2} \cap \Sigma_0$ satisfying $y(p'') < MR_2$. By construction, this p'' must be in $\mathcal{U}_{R_2}$, so there must exist another point p''' on $\partial\mathcal{U}_{R_2}$ whose distance to p' is small. For the fixed constant C defined above, we can further assume that r is small enough such that p' will now sit in $B_{r_3/C} \cap \Sigma_0$ from the point p''', contradicting the assumption that p' is the first point not in $\mathcal{U}^r_{R_2}$.

4.4.4 Tidying up

To complete the proof of the uniqueness theorem, we first use Proposition 4.4.14 to show that $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ vanishes in the component of Σ'_0 that is connected to $\mathfrak{H}_0$. Suppose this component does not cover the entirety of $\Sigma_0 \cap \mathfrak{D}$. We argue now in the same way as Section 3.4 to conclude that any point in $\Sigma_0 \cap \mathfrak{D}$ that does not lie in the desired connected component of Σ'_0 cannot be reached from the interior by any curve of finite Riemannian length, and we obtain a contradiction. Therefore $\mathcal{F}^2 \neq 0$, $4\Xi' \neq 2M$, and $\mathcal{B}'_{ab}$ and $\mathcal{Q}'_{abcd}$ are vanishing in the entirety of $\Sigma_0 \cap \mathfrak{D}$. By stationarity, the conditions hold in the entirety of $\mathfrak{D}$, establishing Theorem 4.4.5.

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